Lecture 01: Groups and symmetries

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Groups are symmetries of objects. Let's see a few examples to understand this sentence better.

- At the level of set theory.

Let X be a set. As X does not have a particular structure any bijection $f: X \to X$ is a symmetry! This group is denoted by S_X, and is called the symmetric group of X.

$$S_X := \{ f: X \to X \mid f \text{ is a bijection} \}.$$

For a positive integer n, we write S_n instead of $S_{\{1,2,\cdots,n\}}$. You have seen before that

$$|S_n| = n! = 1 \times 2 \times \cdots \times n.$$

- Euclidean plane.

Let E be the Euclidean plane; this means as a set $E = \mathbb{R}^2$; but it also has the Euclidean distance.

$$\text{Symmetries of } E = \{ T: E \to E \mid \underbrace{T: \text{bijection}}_{\text{set theory level}};\ T \text{ preserves distance} \}$$

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Euclid characterized all the elements of symm. of E.

He showed any symmetry can be achived as a combination of a translation, a rotation, and/or a reflection about a line.

$\boxed{Q}$ What is the order of a reflection?

$\boxed{A}$ 2

$\boxed{Q}$ What is the order of a translation?

$\boxed{A}$ Infinity (it is NOT a torsion element.)

$\boxed{Q}$ What is the order of a rotation of angle α?

$\boxed{A}$ It depends on α.

A rotation of angle α is torsion; that means it has a finite order $\Longleftrightarrow$ $\alpha/2\pi$ is a rational number (why?)

Using linear algebra, one can write Euclid's result as

Symm. of the Euclidean plane $= \{ T: \mathbb{R}^2 \to \mathbb{R}^2 \mid$
$Tv = Kv + b$ where
K is orthogonal; $K^t K = I$
$b \in \mathbb{R}^2 \}$.

<u>Exercise</u> Prove that any symm. of the Euclidean plane is a

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combination of a translation, a rotation, and/or a reflection.

Hint. The key property is the following rigidity of the Euclidean plane:

Suppose A, B, C are three points that are NOT colinear. Then any point D is uniquely determined by its distance from A, B, and C.

$$D \longmapsto (|AD|, |BD|, |CD|) \quad \text{is a bijection.}$$

(GPS works because of a similar reason.)

This rigidity implies that, if a symmet. ϕ of the Euclidean plane fixes $(0,0)$, $(1,0)$, and $(0,1)$, then ϕ is the identity map.

Now for an arbitrary symm. $\phi: E \to E$, first we compose ϕ with a translation to make sure that $(0,0)$ is fixed; second compose it with a rotation about $(0,0)$ to make sure $(1,0)$ is fixed, too.

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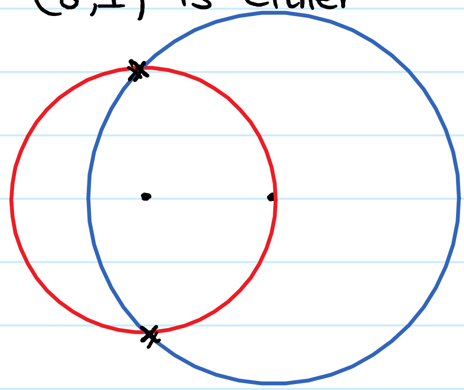
Now that $(0,0)$ and $(1,0)$ are fixed, $(0,1)$ is either sent to itself or to $(0,-1)$.



Hence by composing with a reflection, if needed, we can get that

$$\underbrace{L}_{\text{refle.}} \circ \underbrace{R}_{\text{rotation}} \circ \underbrace{T}_{\text{translation}} \circ \phi$$

fixes the triangle $(0,0), (1,0)$, and $(0,1)$. Therefore it is the identity map.

<u>Symmetries of a graph.</u>

Let $\mathcal{G} = (V, E)$ be a graph. Then the group of symmetries of $\mathcal{G}$ is denoted by $\mathrm{Aut}(\mathcal{G})$;

$$\mathrm{Aut}(\mathcal{G}) = \{ f: V \to V \mid f \text{ is a bijection; } \forall v,w \in V, \{v,w\} \in E \Leftrightarrow \{f(v), f(w)\} \in E \}$$

v is connected to w

$\Updownarrow$

$f(v)$ is connected to $f(w)$.

In many instances, we would like to show that the group of

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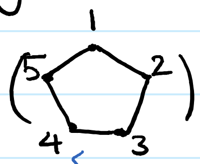
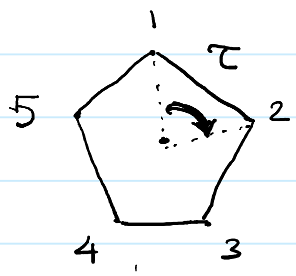
symmetries of an object determines the object in a unique way.

This is how Klein wanted to classify "geometries"; and this point of view is crucial in Galois theory.

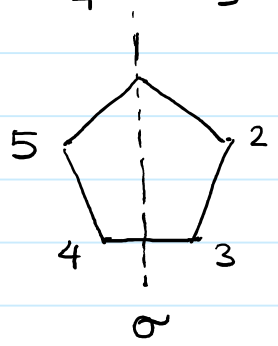
<u>Example</u> Give some elements of Symm()

view it as a graph.



rotation. So $\tau^5 = \mathrm{id}$.

(No fixed point on the graph.)



reflection. So $\sigma^2 = \mathrm{id}$.

(Has exactly one fixed point in the set of vertices)

$\boxed{Q}$ Do σ and τ commute?

$\boxed{A}$ To answer this question we have to look at $\tau\sigma\tau^{-1}$ and find out if it is σ or not. ($\tau\sigma\tau^{-1}$ is called a <u>conjugate of σ</u>; we have <u>conjugated σ by τ</u>.)

A good technique is looking at the fixed point of σ:

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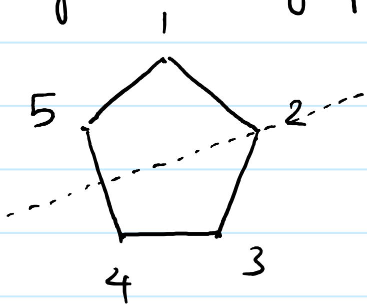
We know $\sigma(1)=1$ and $\tau(1)=2$. So

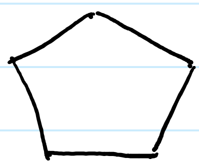
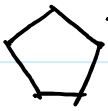
$\sigma(\tau^{-1}(2))=1$, which implies

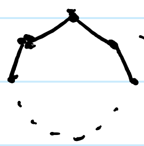
$$(\tau\circ\sigma\circ\tau^{-1})(2)=2.$$

So the fixed point of $\tau\circ\sigma\circ\tau^{-1}$ is different from σ, which implies $\tau\circ\sigma\circ\tau^{-1}$. Looking at the graph, we can see that



$\tau\sigma\tau^{-1}$ can be described $\tau\sigma\tau^{-1}$ as the following reflection:

One can see that if a symmetry of  does not have a fixed point it is a rotation; and if it is not identity and it fixes a point, then it is a "reflection". So $|\mathrm{Symm}($$)|=10$.

<u>Def.</u> Symm() is called the dihedral group D_{2n}.

n-cycle

<u>Exercise</u>. Show that $|D_{2n}|=2n$; n: rotations and n: reflections.

Lecture 01: Group actions

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So far we have started with an object X, and then considered the group of symm. of $X = \{ f: X \to X \mid f:$ bijection and f preserves the structure of $X \}$

Next we would like to make this abstract:

Def. Let G be a group and X be a set. A (left) action of G on X is $m: G \times X \to X$, $m(g, x) = g \cdot x$ which has the following properties:

1. $e \cdot x = x$ for any $x \in X$ where e is the neutral element of G.
2. $\forall x \in X$, $\forall g_1, g_2 \in G$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

we say G acts on X, and write $G \curvearrowright X$.

Important example.

Let X be any object; think about just a set, Euclidean plane, a graph, etc. Then $\mathrm{Symm}(X) \curvearrowright X$.

Pf. $f \in \mathrm{Symm}(X) \Leftrightarrow f: X \to X$ is a bijection and f preserves the structure of X.

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Now we need to define the group action map

$$\mathrm{Symm}(X) \times X \longrightarrow X.$$

$$(f, x) \longmapsto ?$$

The group action should tell us what the group element f does to the point x.

As soon as we phrase the question in this way, we would be forced to think about $f(x)$ as a possible answer. And it is:

Let $m: \mathrm{Symm}(X) \times X \longrightarrow X$, $m(f, x) := f(x)$.

Then $m(I_X, x) = I_X(x) = x$.

the identity function of X is the neutral element of $\mathrm{Symm}(X)$

$\forall f_1, f_2 \in \mathrm{Symm}(X)$, $m(f_1, m(f_2, x)) = m(f_1, f_2(x)) = f_1(f_2(x))$

$$= (f_1 \circ f_2)(x).$$

$$= m(f_1 \circ f_2, x).$$

<u>Ex.</u> $S_n \curvearrowright \{1, 2, \ldots, n\}$; $\underbrace{GL_n(\mathbb{R})}_{n\times n \text{ - invertible matrices}} \curvearrowright \mathbb{R}^n$, $(A, x) \longmapsto Ax$

Lecture 01: Parametrizing group actions

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The following is an important point of view towards <u>functions</u> $G\times X \xrightarrow{m} X$ (here we are not assuming any special property for G, X, or m.)

For any such function, we can fix the first component g and get a function $m_g : X \to X$. This way we get a <u>family</u> $\{m_g\}_{g\in G}$ of functions $m_g : X \to X$.

And this can be reversed:

$$(*) \quad m: G\times X \to X \longmapsto \{m_g\}_{g\in G}, \text{ where } m_g : X \to X$$

$$m(g,x) = m_g(x)$$

is a bijection.

Now we would like to know what happens if $m: G\times X \to X$ is a group action. In the next lecture we will prove

<u>Theorem</u>. There is a bijection between

$\{\text{group action } m: G\times X \to X\}$ and $\mathrm{Hom}(G, S_X)$.

(In fact, the function given in $(*)$ induces a bijection.)