Lecture 03: Recall

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At the end of previous lecture we mentioned

<u>Theorem</u>. Let $A_{G,X} := \{m : G \times X \to X \mid m: \text{group action}\}$.

Then $\Psi : A_{G,X} \to \mathrm{Hom}(G, S_X)$, $\big((\Psi(m))(g)\big)(x) := m(g,x)$,

and $\Phi : \mathrm{Hom}(G, S_X) \to A_{G,X}$, $(\Phi(f))(g,x) := (f(g))(x)$

are inverse of each other.

<u>Outline of proof.</u>

First we fix the group action $\underline{m}$; and for simplicity instead of writing $(\Psi(m))(g) : X \to X$ we simply write $\xi(g)$; and instead of writing $m(g,x)$ we simply write $g \cdot x$.

Then
$$\big(\xi(g_1)\circ\xi(g_2)\big)(x) = \xi(g_1)\big(\xi(g_2)(x)\big)$$
$$= \xi(g_1)(g_2\cdot x) = g_1\cdot(g_2\cdot x)$$
$$= (g_1 g_2)\cdot x = \xi(g_1 g_2)(x).$$

So $\xi(g_1)\circ\xi(g_2) = \xi(g_1 g_2)$. And

$\xi(e)(x) = e\cdot x = x$, which implies $\xi(e) = I_X$.

So $\xi(g)\circ\xi(g^{-1}) = \xi(e) = I_X$ and $\xi(g^{-1})\circ\xi(g) = I_X$.

Therefore $\xi(g) \in S_X$ and $\xi : G \to S_X$ is a group hom.

Lecture 03: Cayley's theorem

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This shows that Ψ is well-defined.

We know that $S_X \curvearrowright X$ via $\sigma \cdot x := \sigma(x)$. So for any $f \in \mathrm{Hom}(G, S_X)$ we get an induced group action $g * x := f(g) \cdot x = f(g)(x)$. This shows that Φ is a well-defined function.

<u>Exercise</u>. Check that $\Psi \circ \Phi = I_{\mathrm{Hom}(G, S_X)}$ and

$$\Phi \circ \Psi = I_{A_{G,X}}.$$

<u>Theorem</u>. Suppose G is a group. Then G can be embedded into the symmetric group S_G of G.

<u>Pf. 1</u>. $G \curvearrowright G$ by the left translation. This action gives us the following group homomorphism from G to S_G:

$\phi : G \to S_G$, $\phi(g) = l_g$ where $l_g : G \to G$, $l_g(g') = gg'$. And as we shall see $\ker \phi = \{e\}$ and so G can be embedded into S_G.

<u>Pf. 2</u>. (This is the same pf, but this time we do not refer to the left translation action. Knowing about action only makes the

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argument a bit more natural.)

For any $g \in G$, let $l_g : G \to G$, $l_g(g') = g g'$.

<u>Step 1.</u> $l_{g_1} \circ l_{g_2} = l_{g_1 g_2}$

<u>Pf of step 1</u> $(l_{g_1} \circ l_{g_2})(g') = l_{g_1}(l_{g_2}(g'))$

$$= l_{g_1}(g_2 g') = g_1(g_2 g') = (g_1 g_2) g' = l_{g_1 g_2}(g').$$

<u>Step 2.</u> $l_e = I_G$ (it is clear)

<u>Step 3.</u> l_g is a bijection.

<u>Pf of step 3</u> $l_g \circ l_{g^{-1}} = l_e = I_G$ and $\Rightarrow l_g$ is invertible

$l_{g^{-1}} \circ l_g = l_e = I_G$ $\Rightarrow l_g$ is a bijection.

Step 3 implies $g \mapsto l_g$ is a function from G to S_G

Step 1 implies the above map is a group hom.

- g is in the kernel of this hom. $\Rightarrow l_g = I_G$

$\Rightarrow l_g(e) = I_G(e) \Rightarrow g \cdot e = e \Rightarrow g = e$.

So G can be embedded into S_G. $\blacksquare$

Lecture 03: Orbits and stabilizers

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Suppose $G \curvearrowright X$. We would like to understand the G-orbits.

Def. The orbit of $x \in X$ is $G\cdot x = \{g\cdot x \mid g \in G\}$.

Lemma. Suppose $G \curvearrowright X$. Then the following are equivalent:

(a) $G\cdot x_1 = G\cdot x_2$

(b) $x_1 \in G\cdot x_2$

(c) $G\cdot x_1 \cap G\cdot x_2 \neq \emptyset$

Pf. (a) $\overset{?}{\Rightarrow}$ (b) $\quad x_1 = e\cdot x_1 \in G\cdot x_1 = G\cdot x_2$.

(b) $\overset{?}{\Rightarrow}$ (c) $\quad x_1 \in G x_2 \Rightarrow x_1 \in G\cdot x_1 \cap G\cdot x_2 \Rightarrow G\cdot x_1 \cap G\cdot x_2 \neq \emptyset$.

(c) $\overset{?}{\Rightarrow}$ (a) Suppse $x \in G\cdot x_1 \cap G\cdot x_2$. Then

$$\exists g_1, g_2 \in G, \quad x = g_1\cdot x_1 = g_2 \cdot x_2.$$

$$\forall g \in G, \quad g\cdot x_1 = g g_1^{-1}\cdot g_1 x_1 = g g_1^{-1} \cdot g_2 \cdot x_2$$

$$= (g g_1^{-1} g_2)\cdot x_2 \in G\cdot x_2$$

So $G\cdot x_1 \subseteq G\cdot x_2$; by a similar argument we have

$G\cdot x_2 \subseteq G\cdot x_1$. Hence $G\cdot x_1 = G\cdot x_2$. ∎

Corollary The set $\{G\cdot x \mid x \in X\}$ of orbits is a partition of X.

Pf. $\forall x \in X$, $x \in G\cdot x \Rightarrow \bigcup_{x\in X} G\cdot x = X$; now Lemma implies the claim.

Lecture 03: The quotient space

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Def. Suppose $G \curvearrowright X$; the set of G-orbits is denoted by $G\backslash X$ and it is called the quotient of X by the G-action.

Example Suppose $H \leq G$. Then $H \curvearrowright G$ by the left translation. Then the H-orbits are exactly the right cosets of H. And $H\backslash G$ is the usual quotient of G by H.

Lemma. Suppose $G \curvearrowright X$. Then, for any $x_0 \in X$,

$$G_{x_0} = \{ g \in G \mid g \cdot x_0 = x_0 \}$$

is a subgroup of G. It is called the stabilizer subgroup of G with respect to x.

Pf. $g_1 \cdot x_0 = x_0$

$g_2 \cdot x_0 = x_0 \Rightarrow g_2^{-1} \cdot (g_2 \cdot x_0) = g_2^{-1} \cdot x_0 \Rightarrow e \cdot x_0 = g_2^{-1} \cdot x_0$

$\Rightarrow g_2^{-1} \cdot x_0 = x_0$

$(g_1 g_2^{-1}) \cdot x_0 = g_1 \cdot (g_2^{-1} \cdot x_0) = g_1 \cdot x_0 = x_0$. ∎

Theorem. (Orbit-Stabilizer theorem) Suppose $G \curvearrowright X$. Then $\forall x \in X$, $G/G_x \xrightarrow{\phi} G \cdot x$, $g G_x \longmapsto g \cdot x$ is a bijection.

Pf. Well-defined and injective: $g_1 G_x = g_2 G_x \Leftrightarrow g_1^{-1} g_2 \in G_x$

Lecture 03: Orbit-Stabilizer theorem

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$$g_1^{-1} g_2 \in G_x \iff (g_1^{-1} g_2)\cdot x = x \iff g_1\cdot((g_1^{-1}g_2)\cdot x) = g_1\cdot x$$

$$\iff g_1\cdot x = g_2 \cdot x.$$

Surjectivity is clear. $\blacksquare$

<u>Def.</u> $G \curvearrowright X$ is called a <u>free action</u> if $\forall x \in X$, $G_x = \{1\}$.

$\boxed{P}$ Give an example of a free action for any given group G.

$\boxed{S}$ $G \curvearrowright G$ by the left translations.

So, if $H \le G$, then, $\forall g \in G$, there is a bijection between H and Hg.

<u>Lagrange's theorem</u> Suppose G is a finite group and $H \le G$. Then $|G| = |{}_H\backslash G|\,|H|$.

<u>Pf.</u> $H \curvearrowright G$ by the left translations and this action is free. Since ${}_H\backslash G$ is a partition of G, we have

$$|G| = \sum \quad |Hg|.$$

$\scriptstyle Hg \in {}_H\backslash G$

Since the action is free, $|Hg| = |H|$ for any $g \in G$.

So $|G| = |{}_H\backslash G|\,|H|$. $\blacksquare$