Lecture 07: Sylow's theorems

Tuesday, October 10, 2017 11:45 PM

<u>Def.</u> Suppose G is a finite group, $p^n \mid |G|$, and $p^{n+1} \nmid |G|$.

Then a subgroup P of order p^n is called a Sylow p-subgp of G. And $\mathrm{Syl}_p(G) = \{P \leq G \mid P \text{ is a Sylow } p\text{-subgp}\}$.

So the 1st Sylow theorem implies $\mathrm{Syl}_p(G) \neq \emptyset$.

- Observe that $G \curvearrowright \mathrm{Syl}_p(G)$ by conjugation.

  <u>Theorem.</u> $G \curvearrowright \mathrm{Syl}_p(G)$ is a transitive action; that means any two Sylow p-subgroups are conjugate.

We prove the following stronger version:

(Sylow's 2nd thm)
<u>Theorem</u> Suppose P' is a p-subgp of G, and $P \in \mathrm{Syl}_p(G)$.

Then $\exists g \in G$, $P' \subseteq g P g^{-1}$.

$P' \curvearrowright G/P$ by the left translations. Since P' is a p-gp,

$$|G/P| \equiv |(G/P)^{P'}| \pmod{p}. \quad \circledast$$

$$gP \in (G/P)^{P'} \iff \forall p' \in P',\ p' g P = gP$$

$$\iff \forall p' \in P',\ g^{-1} p' g \in P$$

$$\iff g^{-1} P' g \subseteq P \iff P' \subseteq g P g^{-1}.$$

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So $P' \subseteq$ a conjugate of $P \iff (G/P)^{P'} \neq \emptyset$.

Since $p \nmid |G/P|$, by ⊛ $p \nmid |(G/P)^{P'}|$. And so $(G/P)^{P'} \neq \emptyset$. ∎

<u>Corollary</u>. Suppose $P \in Syl_p(G)$. Then $Syl_p(N_G(P)) = \{P\}$.

<u>Pf.</u> Since $P \in Syl_p(G)$, $p \nmid |G/P|$. Therefore $p \nmid |N_G(P)/P|$.

So $P \in Syl_p(N_G(P))$. By the previous theorem (Sylow's 2^{nd} theorem)

any Sylow p-subgroup of $N_G(P)$ is a conjugate (in $N_G(P)$)

of P. Since $P \triangleleft N_G(P)$, we deduce $\{P\} = Syl_p(N_G(P))$. ∎

<u>Corollary</u>. Suppose $P \in Syl_p(G)$. Then $N_G(N_G(P)) = N_G(P)$.

<u>Pf.</u> Let $g \in N_G(N_G(P))$. Then by the previous corollary

$$\begin{aligned} \{gPg^{-1}\} &= Syl_p(g\, N_G(P)\, g^{-1}) && \text{(Conjugation by } g \text{ is an automorphism.)} \\ &= Syl_p(N_G(P)) && (g \in N_G(N_G(P)).) \\ &= \{P\} && \text{(previous corollary)} \end{aligned}$$

$$\Rightarrow gPg^{-1} = P \quad \Rightarrow g \in N_G(P).$$

Therefore $N_G(N_G(P)) \subseteq N_G(P)$. The other direction

is clear. ∎

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(Sylow's 3^{rd})

<u>Theorem</u>. $|Syl_p(G)| \equiv 1 \pmod{p}$.

<u>Pf</u>. Let P_0 be a Sylow p-subgroup. If $P_0 = \{1\}$, then

$|Syl_p(G)| = 1$ and we are done. So w.l.o.g. we will assume

$P_0 \neq \{1\}$. $P_0 \curvearrowright Syl_p(G)$ by conjugation. So

$$|Syl_p(G)| \equiv |Syl_p(G)^{P_0}| \pmod{p}. \quad \circledast$$

$$P \in Syl_p(G)^{P_0} \iff \forall p_0 \in P_0,\ p_0 P p_0^{-1} = P$$

$$\iff P_0 \subseteq N_G(P).$$

$$\iff P_0 \in Syl_p(N_G(P)) = \{P\}$$

$$\iff P_0 = P.$$

So $|Syl_p(G)^{P_0}| = 1$.

Therefore by $\circledast$ $|Syl_p(G)| \overset{p}{\equiv} 1$. ∎

Sylow's theorems are very instrumental for describing possible group structures of a group with a given order. Here is a standard example:

<u>Problem</u>. Describe groups of order pq, where p and q are primes, and $p<q$.

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Let $n_q := |\mathrm{Syl}_q(G)|$; and $Q_0 \in \mathrm{Syl}_q(G)$.

Since $G \curvearrowright \mathrm{Syl}_q(G)$ transitively, $|\mathrm{Syl}_q(G)| = |G \cdot Q_0|$

$$= [G : N_G(Q_0)].$$

So $n_q \mid |G/Q_0|$; and, by the 3rd Sylow theorem,

$n_q \equiv 1 \pmod{q}$. Therefore $n_q \mid p$ and $q \mid n_q - 1$.

Since p is prime, either $n_q = 1$ or $n_q = p$. As $p<q$

and $q \mid n_q - 1$, we get that $\underline{n_q = 1}$; this implies $Q_0 \triangleleft G$.