Lecture 10: Schur-Zassenhaus theorem

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<u>Theorem</u> (Schur-Zassenhaus) Suppose $N \lhd G$ and $\gcd(|N|, |G/N|) = 1$. Then $\exists\, H \leq G$ s.t. $|H| = |G/N|$.

<u>Corollary</u>. A short exact sequence $1 \to K \xrightarrow{\phi_1} G \xrightarrow{\phi_2} L \to 1$ splits if $\gcd(|K|, |L|) = 1$.

<u>Pf.</u> Since $1 \to K \xrightarrow{\phi_1} G \xrightarrow{\phi_2} L \to 1$ is a short exact sequence we have $N := \phi_1(K) \lhd G$ and $L \simeq G/\phi_1(K)$.

So by the Schur-Zassenhaus theorem $\exists\, H \leq G$ and $|H| = |G/\phi_1(K)| = |L|$. So $\gcd(|N|, |H|) = \gcd(|K|, |L|) = 1$.

Hence $N \cap H = 1$. Therefore $|NH| = |N||H| = |N|\,|G/N| = |G|$.

So $G = NH$; this implies $G/N = HN/N \simeq H/H\cap N = H$;

and

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] & G \arrow[r] & G/N \arrow[r] \arrow[dl] & 1 \\ & & H \arrow[u, hook] & & \end{tikzcd}$$

splits

Therefore

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "\simeq"] & G \arrow[r] \arrow[d, "="] & L \arrow[r] \arrow[d, "\simeq"] \arrow[l, bend left=60] & 1 \\ 1 \arrow[r] & N \arrow[r] & G \arrow[r, shift left] & G/N \arrow[r] \arrow[l, shift left] & 1 \end{tikzcd}$$

splits. ∎

Lecture 10: Getting to case: N is minimal normal

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Proof of reduction to the case where N is abelian:

We proceed by the strong induction on $|G|$.

Step 1. Suppose N is not a minimal normal subgroup; that means

$\exists\, 1 \neq N_0 \lneq N$ s.t. $N_0 \trianglelefteq G$.

Then consider $N/N_0 \trianglelefteq G/N_0$.

We have $|N/N_0| \,\big|\, |N|$ and $[G/N_0 : N/N_0] = |G/N|$. So

$\gcd(|N/N_0|, [G/N_0 : N/N_0]) = 1$. Therefore by the strong induction

hypothesis, $\exists\, \overline{H} \leq G/N_0$ s.t. $|\overline{H}| = [G/N_0 : N/N_0] = |G/N|$.

$\overline{H} = \widetilde{H}/N_0$ where $\widetilde{H} \leq G$. Now consider $N_0 \trianglelefteq \widetilde{H}$.

1. $$\left.\begin{array}{l} |\widetilde{H}/N_0| = |\overline{H}| = |G/N| \\ |N_0| \,\big|\, |N| \\ \gcd(|N|, |G/N|) = 1 \end{array}\right\} \Rightarrow \gcd(|N_0|, |\widetilde{H}/N_0|) = 1$$

2. $$\left.\begin{array}{l} |\widetilde{H}| = \dfrac{|N_0|}{|N|}|G| \\ |N_0| < |N| \end{array}\right\} \Rightarrow |\widetilde{H}| < |G|$$

So by the strong induction hypothesis, $\exists\, H \leq \widetilde{H}$ s.t.

$|H| = |\widetilde{H}/N_0| = |G/N|$.

Lecture 10: Getting to case: N is p-group

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Using Step 1, from this point on, we will assume N is a minimal normal subgroup of G.

Step 2. Suppose $p \mid |N|$, but N is not a p-group.

Let P be a Sylow p-subgroup. Then as you saw in your homework assignment $G = N\, N_G(P)$.

So $G/N \simeq N_G(P)/_{N \cap N_G(P)}$

Consider $N \cap N_G(P) \triangleleft N_G(P)$.

① Since $P \lneq N$ and N is a minimal normal subgp of G, $P \ntriangleleft G$.

So $N_G(P) \neq G$. $\Rightarrow |N_G(P)| < |G|$.

② $|N \cap N_G(P)| \mid |N|$

$|N_G(P)/_{N \cap N_G(P)}| = |G/N|$

$\gcd(|N|, |G/N|) = 1$

$\Rightarrow \gcd(|N \cap N_G(P)|, |N_G(P)/_{N \cap N_G(P)}|) = 1$.

So by the strong induction hypothesis, $\exists\, H \leq N_G(P)$ s.t.

$|H| = |N_G(P)/_{N \cap N_G(P)}| = |G/N|$. ■

Using Step 1 and Step 2, we can and will assume:

$N = P$ is a p-group, and it is a minimal normal subgroup of G.

Lecture 10: Getting to case: N is abelian

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Now we prove under the above conditions $N=P$ is abelian;

We have proved that the center $Z(P)$ of a p-group is non-trivial.

<u>Claim.</u> $Z(P) \trianglelefteq G$.

<u>Pf of claim.</u> $\forall g \in G$, conjugation by g is an automorphism of G. So $g Z(P) g^{-1} = Z(gPg^{-1})$.

Since $P \trianglelefteq G$, we have $gPg^{-1} = P$. Therefore

$$g Z(P) g^{-1} = Z(P). \quad \blacksquare$$

Since $1 \neq Z(P) \leq P$, $Z(P) \trianglelefteq G$, and P is a minimal normal subgroup, we have $Z(P) = P$; this means P is abelian. $\blacksquare$

The abelian case you will do as part of your homework assignment.

We have also recalled: $\begin{array}{l} G \xrightarrow{\pi} G/N \\ g \longmapsto gN \end{array}$ is the canonical quotient map; Suppose $\overline{H} \leq G/N$. Then the preimage $\pi^{-1}(\overline{H})$ (let's call it H) is a subgp of G; and $\pi^{-1}(1) \trianglelefteq H$; this implies $N \trianglelefteq H$. And $\overline{H} = H/N$ as π is surjective.