Lecture 15: Jordan-Holder theorem

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Simple groups can be treated as building blocks of (finite) groups.

<u>Def.</u> A <u>composition series</u> of a group G is

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \cdots \trianglelefteq N_k = G$$

such that N_i/N_{i-1} is a simple group for any integer $i \in [1, k]$.

For any i, N_i/N_{i-1} is called a composition factor of G.

<u>Def.</u> We write $(S_1, \dots, S_m) \sim (S'_1, \dots, S'_m)$ if they are the same sequence after a rearranging. (This is only an auxil. notation.)

<u>Theorem.</u> (Jordan-Hölder) Suppose G is a finite group; $|G| > 1$.

(a) G has a composition series; that means there are subgps

$$\{1\} = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_k = G \text{ s.t. } N_i/N_{i-1} \text{ are simple.}$$

(b) If $\{1\} = M_0 \triangleleft M_1 \triangleleft \cdots \triangleleft M_s = G$ and $\{1\} = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_r = G$ are two composition series, then $r = s$ and

$$(M_1/M_0, \dots, M_s/M_{s-1}) \sim (N_1/N_0, \dots, N_r/N_{r-1}).$$

<u>Pf.</u> (a) Among all chains: $\{1\} = N_0 \underset{\neq}{\trianglelefteq} \cdots \underset{\neq}{\trianglelefteq} N_k = G$; take a longest one.

Notice that, since G is a non-trivial finite group, there is such a

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chain.

<u>Claim</u> For any i, N_i/N_{i-1} is simple.

<u>Pf.</u> if not, $\exists\ \{\bar{e}\} \neq \bar{M} \underset{\neq}{\trianglelefteq} N_i/N_{i-1}$. Let $\pi: N_i \to N_i/N_{i-1}$ be the quotient map; and $M := \pi^{-1}(\bar{M})$. Then

$$\begin{array}{ccccc} \pi^{-1}(\{\bar{e}\}) & \underset{\neq}{\triangleleft} & \pi^{-1}(\bar{M}) & \underset{\neq}{\trianglelefteq} & \pi^{-1}(N_i/N_{i-1}) \\ \| & & \| & & \| \\ N_{i-1} & \underset{\neq}{\triangleleft} & M & \underset{\neq}{\trianglelefteq} & N_i \end{array}$$

So $e = N_0 \underset{\neq}{\trianglelefteq} \cdots \underset{\neq}{\trianglelefteq} N_{i-1} \underset{\neq}{\trianglelefteq} M \underset{\neq}{\trianglelefteq} N_i \underset{\neq}{\trianglelefteq} \cdots \underset{\neq}{\trianglelefteq} N_k = G$ is a longer chain; this is a contradiction.

(b) We proceed by strong induction on $|G|$. The base case, $|G| = 2$, is clear. Suppose $\{e\} =: N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_r := G$ and

$$\{e\} =: M_0 \triangleleft M_1 \triangleleft \cdots \triangleleft M_s := G$$

are two composition series.

<u>Case 1.</u> If $N_{r-1} = M_{s-1}$, then using the strong induction hypothe. for N_{r-1}, we get (1) $r-1 = s-1$

$$(2)\ (N_1/N_0, \ldots, N_{r-1}/N_{r-2}) \sim (M_1/M_0, \ldots, M_{s-1}/M_{s-2})$$

Now (1), (2), and $M_s/M_{s-1} = N_r/N_{r-1}$ imply the claim.

<u>Case 2.</u> Suppose $N_{r-1} \neq M_{s-1}$. Let $N := N_{r-1}$ and $M := M_{s-1}$.

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Then $\underset{M\neq N}{M,N \triangleleft G}$; and so $M \neq MN \triangleleft G \Rightarrow e \neq MN/M \triangleleft G/M$.

Since G/M is simple, we deduce that $G = MN$. Therefore

(1) $\boxed{G/M = MN/M \simeq N/M\cap N \quad \text{and} \quad G/N = MN/N \simeq M/M\cap N.}$

In particular, these are simple groups.

Let $\{e\} = K_0 \triangleleft K_1 \triangleleft \cdots \triangleleft K_t := M\cap N$ be a composition series.

(1) implies that $\{e\} = K_0 \triangleleft \cdots \triangleleft K_t \triangleleft M$ and
$\{e\} = K_0 \triangleleft \cdots \triangleleft K_t \triangleleft N$ are

composition series. Next we will use the strong induction hyp. for M and N.

Notice that $M_0 \triangleleft M_1 \triangleleft \cdots \triangleleft M_{s-1} = M$ and
$K_0 \triangleleft K_1 \triangleleft \cdots \triangleleft K_t \triangleleft M$

are two composition series of M. And so by the strong induction hypothesis, we have

(2)
$$\boxed{\begin{aligned} & t = s-1, \text{ and} \\ & (M_1/M_0, \ldots, M_{s-1}/M_{s-2}) \sim (K_1/K_0, \ldots, K_t/K_{t-1}, M/K_t) \end{aligned}}$$

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Similarly using $N_0 \lhd N_1 \lhd \cdots \lhd N_{r-1} = N$ and

$K_0 \lhd K_1 \lhd \cdots \lhd K_t \lhd N$,

and the strong induction hypothesis, we deduce

(3) $\boxed{t = r-1, \text{ and } (N_1/N_0, \ldots, N_{r-1}/N_{r-2}) \sim (K_1/K_0, \ldots, K_t/K_{t-1}, N/K_t)}$

(2), (3) $\Rightarrow$ $s = r$.

(1) $\Rightarrow$

$$(K_1/K_0, \ldots, K_t/K_{t-1}, N/K_t, G/N_{r-1}) \sim (K_1/K_0, \ldots, K_t/K_{t-1}, M/K_t, G/M_{s-1})$$

because of (3) $\wr$ $\qquad\qquad$ $\wr$ because of (2)

$$(N_1/N_0, \ldots, N_{r-1}/N_{r-2}, G/N_{r-1}) \qquad (M_1/M_0, \ldots, M_{s-1}/M_{s-2}, G/M_{s-1})$$

and the claim follows. ∎

<u>Ex.</u> Let A be a finite abelian group of order $\prod p_i^{k_i}$ where p_i's are distinct primes. Then the composition factors of A are k_i-times $\mathbb{Z}/p_i\mathbb{Z}$.

<u>Pf</u>. Since A is abelian, all the composition factors are abelian. So they are cyclic groups of prime order. If S is

a composition factor, then $|S| \mid |A|$. So $|S| = p_i$ for some i. On the other hand, if $(S_1, \ldots, S_m)$ are composition factors of A, then $|A| = \prod_{i=1}^{m} |S_i|$; and using unique factorization to primes, claim follows. ∎

In the next lecture we will define solvable groups, and we will see that a finite group G is solvable if and only if its composition factors are cyclic of prime order.