Lecture 18: Finite nilpotent groups

Monday, November 6, 2017 10:58 AM

In the previous lecture we proved: $H \subsetneq G \Rightarrow H \subsetneq N_G(H)$ if G is nilpotent.

<u>Corollary</u>. Suppose G is a finite nilpotent gp.; and $P \in Syl_p(G)$.

Then $P \trianglelefteq G$.

<u>Pf</u>. We have seen that $N_G(N_G(P)) = N_G(P)$. So by the previous Proposition, $N_G(P) = G$; this means $P \trianglelefteq G$. ∎

<u>Proposition</u>. Suppose G is a finite nilpotent group. Then $\forall p \mid |G|$, there is a unique Sylow p-subgp P_p and

$$G \simeq \prod_{p \mid |G|} P_p .$$

<u>Pf</u>. We have already proved that any Sylow p-subgp is normal. Since Sylow p-subgps are conjugate, there is a unique Sylow p-subgp P_p. So using the following lemma, one can deduce the claim.

<u>Lemma</u>. Suppose $N_1, \ldots, N_k \trianglelefteq G$ and $\gcd(|N_i|, |N_j|) = 1$. Then

$$N_1 \times N_2 \times \cdots \times N_k \simeq N_1 \cdot N_2 \cdot \cdots \cdot N_k$$
$$(x_1, x_2, \ldots, x_k) \mapsto x_1 \cdot x_2 \cdot \cdots \cdot x_k$$

<u>Pf</u>. We proceed by induction on k. The base case is trivial.

<u>The induction step</u>. By the induction hypothesis, $N_1 \times \cdots \times N_k \simeq N_1 \cdots N_k$

$$(x_1, \ldots, x_k) \mapsto x_1 \cdots x_k$$

In particular, $|N_1 \cdots N_k| = \prod_{i=1}^{k} |N_i|$. Hence $\gcd(|N_{k+1}|, |N_1 \cdots N_k|) = 1$.

Let $N := N_1 \cdots N_k$. So $N \cap N_{k+1} = 1$ and $N \trianglelefteq G$.

$N \trianglelefteq G, N_{k+1} \trianglelefteq G \Rightarrow [N, N_{k+1}] \subseteq N \cap N_{k+1} = 1$. Therefore N comm. with N_{k+1}. Let $\phi : N \times N_{k+1} \to G$, $\phi(x, x_{k+1}) = x x_{k+1}$. Then

- $\phi((x, x_{k+1})(x', x'_{k+1})) = \phi(xx', x_{k+1} x'_{k+1}) = xx' x_{k+1} x'_{k+1}$

$$= x x_{k+1} x' x'_{k+1} = \phi(x, x_{k+1}) \phi(x', x'_{k+1}).$$

- $\phi(x, x_{k+1}) = 1 \Rightarrow x x_{k+1} = 1 \Rightarrow x = x_{k+1}^{-1} \in N \cap N_{k+1} = 1$

$$\Rightarrow (x, x_{k+1}) = (1, 1).$$

$\operatorname{Im} \phi = N \cdot N_{k+1}$; and the claim follows. ∎

<u>Proposition</u>. A finite p-gp P is nilpotent.

<u>Pf</u>. If $Z_i(P) \neq P$, then $P/Z_i(P)$ is a non-trivial finite p-gp. So $Z(P/Z_i(P))$ is non-trivial; this implies $Z_i(P) \lneq Z_{i+1}(P)$.

Since P is finite, $\exists c$ s.t. $Z_c(P) = P$. ∎

<u>Proposition</u>. Suppose $G_1, \ldots, G_k$ are nilpotent gps. Then

$G_1 \times \cdots \times G_k$ is nilpotent.

<u>Exerc</u>. By induction on l show that

$$\gamma_l(G_1 \times \cdots \times G_k) = \gamma_l(G_1) \times \cdots \times \gamma_l(G_k).$$

Lecture 18: Some properties of nilpotent groups

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Theorem. A finite group G is nilpotent if and only if it is a direct product of p-groups.

(This is a summary of the previous propositions.)

Proposition. A finite group is nilpotent if and only if every maximal subgp is normal.

Pf. ($\Rightarrow$) Suppose $M \lneq G$ is a maximal subgp. Since G is nilpotent $N_G(M) \gneq M$. As M is a maximal subgroup, we get that $N_G(M) = G$; this means $M \trianglelefteq G$.

($\Leftarrow$) It is enough to prove that any Sylow p-subgp is normal (why?). So let P be a Sylow p-subgp; and suppose to the contrary that $P \ntrianglelefteq G$. Hence $N_G(P)$ is a proper subgp. Therefore there is a maximal subgp M which contains $N_G(P)$ as a subgp. By the assumption $M \trianglelefteq G$. And $P \leq M$ is a Sylow p-subgp of M. So $G = N_G(P).M$; this is a contradiction as $N_G(P).M = M$.

$\blacksquare$