# Lecture 19: Some properties of nilpotent groups

Monday, November 6, 2017 8:44 AM

**Proposition.** Suppose $G$ is nilpotent. Then

① $G$ is solvable.

② If $1 \neq N \trianglelefteq G$, then $Z(G) \cap N \neq 1$; in particular $Z(G) \neq 1$ if $G \neq 1$.

③ $H \leq G$, $N \trianglelefteq G \Longrightarrow H$ and $G/N$ are nilpotent.

**Pf.** ① By induction on $i$, we have $\gamma_i(G) \supseteq G^{(i)}$.

(In fact using one of your HW assignments you can prove $G^{(i)} \subseteq \gamma_{2^i}(G)$.)

② Suppose $\gamma_i(G) \cap N \neq 1$ and $\gamma_{i+1}(G) \cap N = 1$. There is such an $i$ as $\gamma_1(G) \cap N = N \neq 1$ and $\gamma_{1+c}(G) \cap N = 1$.

Then $[\gamma_i(G) \cap N, N] \subseteq \gamma_{i+1}(G) \cap N = 1$.

So $\gamma_i(G) \cap N \subseteq Z(N)$.

③ By induction on $i$, show $\gamma_i(H) \subseteq \gamma_i(G)$ and $\gamma_i(G/N) = \gamma_i(G)N/N$. □

# Lecture 19: Frattini subgroup of a p-group

Monday, November 13, 2017 8:48 AM

Def. The Frattini subgp $\Phi(G)$ of a group is the intersection of all of its maximal subgps.

Observation. $\forall\, \theta \in \mathrm{Aut}(G)$ and $M < G$ maximal, we have $\theta(M)$ is a maximal subgp. So $\Phi(G)$ is a characteristic subgp of $G$.

Theorem. Suppose $G$ is a finite p-gp. Then

$$\Phi(G) = G^p[G,G], \text{ where } G^p = \{g^p \mid g \in G\}.$$

(Notice that $G^p$ is not necessarily a subgp.)

Pf. Suppose $M$ is a maximal subgp of $G$. Since $G$ is a finite p-gp, it is nilpotent; and so $M \trianglelefteq G$. Therefore $G/M$ is a group with no proper non-trivial subgp; therefore $G/M$ has prime order. Since $G$ is a p-gp, $G/M \simeq \mathbb{Z}/p\mathbb{Z}$. Hence $[G,G] \subseteq M$ as $G/M$ is abelian; and $G^p \subseteq M$ as $G/M$ is p-torsion; that means $(gM)^p = M$. Therefore $G^p[G,G] \subseteq M$. We have proved that $G^p[G,G] \subseteq \Phi(G)$. 1

# Lecture 19: Frattini subgroup of finite p-groups

Wednesday, November 15, 2017 8:43 AM

On the other hand, $G/[G,G]$ is an abelian group; and so $(G/[G,G])^p$ is a normal subgroup. And

$$(G/[G,G])^p = \{(g[G,G])^p \mid g \in G\}$$
$$= \{g^p[G,G] \mid g \in G\} = G^p[G,G]/[G,G].$$

So $G^p[G,G]$ is a normal subgroup of $G$; and $V_0 := G/G^p[G,G]$ is a $p$-torsion abelian gp. Hence $G/G^p[G,G]$ is a vector space over the finite field $\mathbb{Z}/p\mathbb{Z}$. (why?)

Hence for any non-zero vector $v$, there is a subspace $V$ of codimension 1 s.t. $v \notin V$. Hence $V_0/V \simeq \mathbb{Z}/p\mathbb{Z}$ and $v \notin V$. So $V$ is a maximal subgp of $V_0$.

Suppose $g \in G \setminus G^p[G,G]$; then $v := gG^p[G,G] \in V_0$ and $v \neq 0$. Now let $V$ be as above. So $V = M/G^p[G,G]$ for some maximal subgp $M$ of $G$; and $g \notin M$. That means $g \notin \Phi(G)$.

We have proved $g \notin G^p[G,G] \Rightarrow g \notin \Phi(G)$.

So $\Phi(G) \subseteq G^p[G,G]$. ②

Claim follows from ①, ②. ■

# Lecture 19: Frattini subgroups under a homomorphism

Wednesday, November 15, 2017 11:13 AM

Here is an alternative way to explain the $2^{nd}$ part of the proof.

<u>Lemma</u>. Suppose $\theta : G \to H$ is an onto group homomorphism. Then

$$\theta(\Phi(G)) \subseteq \Phi(H).$$

<u>Pf.</u> Suppose $M$ is a maximal subgroup of $H$.

<u>Claim</u>. $\theta^{-1}(M)$ is a maximal subgp of $G$.

<u>Pf of claim</u>. Since $\theta$ is a group hom, $\theta^{-1}(M)$ is a subgp.

- Since $\theta$ is onto and $M$ is a proper subgp, $\theta^{-1}(M)$ is a proper subgp.
- Suppose $\theta^{-1}(M) \lneq \tilde{M} \leq G$.

<u>Subclaim</u>. $\theta^{-1}(\theta(\tilde{M})) = \tilde{M}$.

<u>Pf of subclaim</u>. • $\tilde{M} \subseteq \theta^{-1}(\theta(\tilde{M}))$ is true for any function $\theta$.

• $x \in \theta^{-1}(\theta(\tilde{M})) \Rightarrow \theta(x) \in \theta(\tilde{M}) \Rightarrow \exists\, \tilde{m} \in \tilde{M},\ \theta(x) = \theta(\tilde{m})$
$\Rightarrow \theta(\tilde{m}^{-1}x) = 1 \Rightarrow \tilde{m}^{-1}x \in \theta^{-1}(\{1\}) \subseteq \theta^{-1}(M) \subseteq \tilde{M}$
$\Rightarrow x \in \tilde{m} \cdot \tilde{M} = \tilde{M}$. ■

Then $\theta^{-1}(M) \lneq \theta^{-1}(\theta(\tilde{M}))$. So $M \lneq \theta(\tilde{M}) \leq H$. Since $M$ is max., we deduce that $\theta(\tilde{M}) = H$ and so $\tilde{M} = \theta^{-1}(\theta(\tilde{M})) = G$. ■

# Lecture 19: Frattini subgroups of finite p-groups

Wednesday, November 15, 2017 11:32 AM

So $\bigcap_{\substack{M < H \\ \text{max.}}} \theta^{-1}(M) \supseteq \Phi(G)$. Hence

$$\theta(\Phi(G)) \subseteq \theta\Big(\bigcap_{\substack{M<H \\ \text{max.}}} \theta^{-1}(M)\Big) \underset{\boxed{\theta \text{ is onto}}}{=} \bigcap_{\substack{M<H \\ \text{max}}} M = \Phi(H). \quad \blacksquare$$

<u>Lemma</u>. Suppose $V$ is a vector space over $\mathbb{Z}/p\mathbb{Z}$. Then

$$\Phi(V) = \{0\}.$$

(Ex.)

---

- $G/G^p[G,G]$ is a vector space over $\mathbb{Z}/p\mathbb{Z}$ $\Rightarrow$ $\Phi(G/G^p[G,G])$ is trivial.
- $\pi : G \to G/G^p[G,G]$ is onto. So $\pi(\Phi(G)) \subseteq \Phi(G/G^p[G,G])$

$\Phi(G) \subseteq \ker \pi = G^p[G,G]$.