Lecture 21: Universal property of free groups

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In the previous lecture we defined the free product of a family $\{G_i\}_{i\in I}$ of groups and proved that

$$\mathrm{Hom}(\ast_{i\in I} G_i, G) \longrightarrow \prod_{i\in I} \mathrm{Hom}(G_i, G)$$
$$f \longmapsto (f|_{G_i})_{i\in I}$$

is a bijection.

Then we defined the free group generated by X:

$$F(X) := \ast_{x\in X} \mathbb{Z}.$$

Universal property of free groups.

Suppose G is a group, and $f: X \to G$ is a function; then $\exists!\ \hat{f}: F(X) \to G$ which is a group homo. and $\hat{f}|_X = f$.

Pf. $\forall x \in G$, let $f_x: \mathbb{Z} \to G$, $f_x(n) := f(x)^n$. Then $f_x \in \mathrm{Hom}(\mathbb{Z}, G)$. So $\exists\ \hat{f} \in \mathrm{Hom}(\ast_{x\in X} \mathbb{Z}, G)$ s.t. $\hat{f}|_{\text{the } x\text{-copy}} = f_x$; and so $\hat{f}(x) = f_x(x) = f(x)$ $\forall x \in X$.

(uniqueness is clear.) ■

Reduced words in a free product.

Suppose $\{G_i\}_{i\in I}$ is a family of finite gps. Then a word

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in the language $\mathcal{L}(X)$ where X is the disjoint union of G_i's is called reduced if $\omega = x_1 \cdot x_2 \cdot \dots \cdot x_n$ and x_i, x_{i+1} are not in the same G_j for some j and x_i is NOT the neutral element of one of the G_j's.

- A word in $\mathcal{L}(X \cup X^{-1})$ is called reduced if $\omega = x_1 \cdot x_2 \cdot \dots \cdot x_n$, $x_i \neq x_{i+1}^{-1}$ and $x_i^{-1} \neq x_{i+1}$.
- A word in $\mathcal{L}(X \cup X^{-1})$ is called cyclically reduced if $\omega = x_1 \cdot x_2 \cdot \dots \cdot x_n$ is reduced and $x_1 \neq x_n^{-1}$ and $x_1^{-1} \neq x_n$.

Lemma. (1) Any element of $\ast_{i \in I} G_i$ can be represented by a reduced word.

(2) Any element of the free group $F(X)$ generated by X can be represented by a reduced word.

(3) Any element of the free gp $F(X)$ gen. by X has a conjugate which can be represented by a cyclically reduced word.

(try to prove this.)

Lecture 21: Examples; and a basic property

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<u>Ex.</u> Suppose F is freely generated by x and y. Then

- $x \cdot x \cdot y$ is a reduced word in $\mathcal{L}(\{x, y, x^{-1}, y^{-1}\})$, but
- $x \cdot x^{-1} \cdot y$ is NOT.
- $x \cdot y \cdot x^{-1}$ is a reduced word, but it is not cyclically reduced.
- $x \cdot x \cdot y$ as an element of $\langle x \rangle * \langle y \rangle$ is NOT considered reduced.

<u>Lemma</u>. Any group G is a quotient of a free group.

<u>Pf.</u> Let $F(G)$ be the free group gen. by the <u>set</u> G. And let $f: G \to G$ be the identity map. So $\exists! \ \hat{f}: F(G) \to G$ which is a group hom. and $\hat{f}|_G = f$. Hence $\hat{f}$ is onto. Therefore $G \simeq F(G)/\ker \hat{f}$; and the claim follows. ∎

. There are many questions about $\mathrm{Aut}(F_n)$ that are still open. They are attacked using combinatorial, topological, or Lie theoretical techniques.
(combinatorial alg.) (geom. gp theory)

Lecture 21: Ping-pong lemma

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<u>Ping-pong lemma</u> Suppose $G \curvearrowright X$, $G_1, G_2 \le G$, $|G_1| \ge 3$, $|G_2| \ge 2$; $X_1, X_2 \subseteq X$ and $X_1 \not\subseteq X_2$ and $X_2 \not\subseteq X_1$. Suppose $(G_2 \setminus 1) \cdot X_1 \subseteq X_2$ and $(G_1 \setminus 1) \cdot X_2 \subseteq X_1$. Then

$$\langle G_1 \cup G_2 \rangle \simeq G_1 * G_2 .$$

<u>Pf.</u> Let $H := \langle G_1 \cup G_2 \rangle$, $f_i : G_i \hookrightarrow H$. Then

$\exists!\ \hat{f} : G_1 * G_2 \to H$, $\hat{f} : G_i \hookrightarrow H$.
group hom.
(and so $\operatorname{Im} \hat{f} = H$).

If $\ker \hat{f} \ne 1$, then $\exists$ a non-trivial reduced word that is sent to 1.

<u>Case 1.</u> $\omega = a_1 b_1 a_2 b_2 \cdots a_{n-1} b_{n-1} a_n$ where $\underset{\ne 1}{a_i} \in G_1$ and $\underset{\ne 1}{b_i} \in G_2$.

Then $\hat{f}(\omega) \cdot X_2 = a_1 b_1 a_2 \cdots b_{n-1} a_n \cdot X_2$

$= a_1 b_1 \cdots b_{n-1} X_1 = a_1 b_1 \cdots a_{n-2} \cdot X_2$

$\cdots$

$= X_1$. So $\hat{f}(\omega) \ne 1$ which is a contradiction.

<u>Case 2.</u> $\omega = b_1 a_1 b_2 \cdots b_{n-1} a_{n-1} b_n$ is similar.

<u>Case 3.</u> $\omega = a_1 b_1 \cdots a_n b_n$; $\exists\, a \in G_1 \setminus \{1, a_1\}$, and consider

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$$a^{-1}\omega a = \underbrace{(a^{-1}a_1)}_{\neq 1} b_1 a_2 b_2 \cdots a_{n-1} b_{n-1} a.$$

So by case 1, $\hat{f}(a^{-1}\omega a) \neq 1$; therefore $\hat{f}(\omega) \neq 1$.

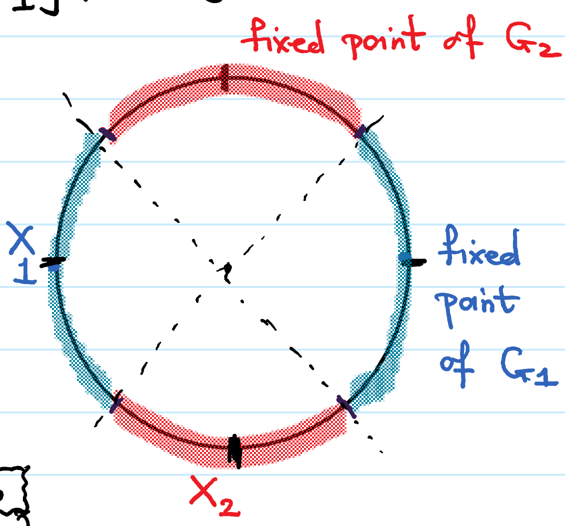
<u>Case 4.</u> $\omega = b_1 a_1 b_2 a_2 \cdots b_n a_n$; consider $a\omega a^{-1}$ where $a \notin \{1, a_n\}$ and argue as in case 3. ∎

<u>Ex.</u> $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ freely generate a subgp of $SL_2(\mathbb{Z})$.

<u>Pf.</u> $G_1 := \left\langle \begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 1 & 2n \\ & 1 \end{bmatrix} \middle| n \in \mathbb{Z} \right\} \simeq \mathbb{Z}$

and $G_2 := \left\langle \begin{bmatrix} 1 & \\ 2 & 1 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 1 & \\ 2n & 1 \end{bmatrix} \middle| n \in \mathbb{Z} \right\} \simeq \mathbb{Z}$.

Let $G := \left\langle \begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & \\ 2 & 1 \end{bmatrix} \right\rangle$.



$G \curvearrowright \mathbb{P}(\mathbb{R}^2)$.

$X_1 := \{ [x:y] \mid |y| \leq |x| \}$

$X_2 := \{ [x:y] \mid |x| \leq |y| \}$

$$\begin{bmatrix} 1 & 2n \\ & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2ny \\ y \end{bmatrix}$$

$$|y| \geq |x| \Rightarrow |x + 2ny| \geq |2n||y| - |x| \underset{\boxed{n \neq 0}}{\geq} 2|y| - |x| \geq |y|$$

And so $[x+2ny : y] \in X_1$.

$$\begin{bmatrix} 1 & \\ 2n & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2nx + y \end{bmatrix}.$$ If $|x| \geq |y|$, and $n \neq 0$ then $|2nx + y| \geq |x|$.

And so by the ping-pong lemma, $G \simeq G_1 * G_2 \simeq \mathbb{Z} * \mathbb{Z} \simeq F_2$ (free group in two generators.) ∎