Lecture 22: Applications of the ping-pong lemma

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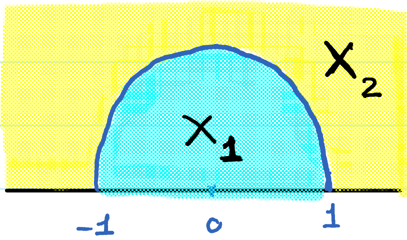
Ex. $\left\langle \overline{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}, \overline{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \right\rangle \simeq \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$; where $\bar{g}$ means

$$g\, Z(SL_2(\mathbb{R})) \in SL_2(\mathbb{R}) / \underbrace{Z(SL_2(\mathbb{R}))}_{\{I, -I\}} =: PSL_2(\mathbb{R})$$

Pf. This time we use the action of $SL_2(\mathbb{R})$ on the upper-half plane; $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \dfrac{az+b}{cz+d}$.

And so $\begin{bmatrix} 1 & 2n \\ & 1 \end{bmatrix} \cdot z = z + 2n$ and

$$\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \cdot z = \frac{-1}{z}$$



Therefore $\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$ sends the blue region to the yellow region, and the yellow region to the blue region.

A shift by at least two steps send X_1 to X_2. So we have

$$\left(\left\langle \begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix} \right\rangle \setminus I \right) \cdot X_1 \subseteq X_2$$

$$\left(\left\langle \overline{\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}} \right\rangle \setminus I \right) \cdot X_2 \subseteq X_1 .$$

Thus by the ping-pong lemma

$$\left\langle \overline{\begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix}}, \overline{\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}} \right\rangle \simeq \left\langle \overline{\begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix}} \right\rangle * \left\langle \overline{\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}} \right\rangle \simeq \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}.$$

Lecture 22: Examples of Schottky groups

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Ex. Suppose $\lambda > 1$, and let $a = \begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}$. Suppose $b \in SL_2(\mathbb{R})$ has the following property:

$$b \cdot \{0, \infty\} \cap \{0, \infty\} = \emptyset. \quad \text{Then}$$

$$SL_2(\mathbb{R}) \curvearrowright \mathbb{R} \cup \{\infty\}, \quad \begin{bmatrix} x & y \\ z & t \end{bmatrix} \cdot r := \frac{xr + y}{zr + t}.$$

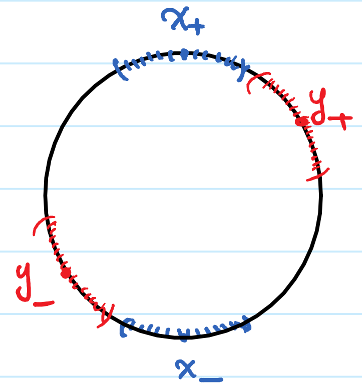
Notice that a has exactly two fixed points: $a \cdot 0 = 0$, $a \cdot \infty = \infty$; and a is contracting everything except 0 towards ∞.

$$\mathrm{Fix}(bab^{-1}) = b \cdot \mathrm{Fix}(a) = \{b \cdot 0, b \cdot \infty\};$$

and so $\mathrm{Fix}(bab^{-1}) \cap \mathrm{Fix}(a) = \emptyset$.

- a is contracting everything except 0 towards ∞
- a^{-1} is contracting everything except ∞ towards 0
- bab^{-1} is contracting everything except $b \cdot 0$ towards $b \cdot \infty$
- $ba^{-1}b^{-1}$ is contracting everything except $b \cdot \infty$ towards $b \cdot 0$

Let's use circle model of $\mathbb{R} \cup \{\infty\}$.



Suppose $a_1, a_2 \in \mathrm{Homeo}(S^1)$;

a_1 has two fixed points x^- and x^+. And

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there are nbhds U_x^- of x^- and U_x^+ of x^+ s.t.

$$a_1^n \cdot (S^1 \setminus U_x^-) \subseteq U_x^+ \quad \forall\, n \in \mathbb{Z}^+$$

$$a_1^{-n} \cdot (S^1 \setminus U_x^+) \subseteq U_x^- \quad \forall\, n \in \mathbb{Z}^+$$

a_2 has two fixed points y^- and y^+ And

there are nbhds U_y^- of y^- and U_y^+ of y^+ s.t.

$$a_2^n \cdot (S^1 \setminus U_y^-) \subseteq U_y^+ \quad \forall\, n \in \mathbb{Z}^+$$

$$a_2^{-n} \cdot (S^1 \setminus U_y^+) \subseteq U_y^- \quad \forall\, n \in \mathbb{Z}^+$$

Suppose $U_x^{\pm}$ and $U_y^{\pm}$'s are disjoint.

Let $X_1 := U_x^+ \cup U_x^-$ and $X_2 := U_y^+ \cup U_y^-$.

Then $$(\langle a_1 \rangle \setminus I) \cdot X_2 \subseteq X_1$$

and $(\langle a_2 \rangle \setminus I) \cdot X_1 \subseteq X_2$. And so by the ping-pong lemma $\langle a_1, a_2 \rangle \simeq \langle a_1 \rangle * \langle a_2 \rangle \simeq \mathbb{Z} * \mathbb{Z}$. So:

<u>Theorem</u>. Suppose $a = \begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}$ where $\lambda > 1$; and $b \in SL_2(\mathbb{R}) \setminus \left(\begin{bmatrix} * & * \\ & * \end{bmatrix} \cup \begin{bmatrix} * & \\ * & * \end{bmatrix} \cup \begin{bmatrix} & * \\ * & \end{bmatrix} \right)$. Then for large enough n, $\langle a^n, b a^n b^{-1} \rangle \simeq F_2$. In particular, $\langle a, b \rangle$ has a

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non-commutative free subgroup.

<u>Remark</u>. The conditions on b are necessary as otherwise $\langle a, b\rangle$ has a subgp of index ≤ 2 which is solvable.

<u>Theorem</u> (Jacques Tits) A finitely generated subgp Γ of $GL_n(F)$ (where F is a field) is either virtually solvable or it contains a non-commutative free subgroup.

. We say Γ is <u>virtually solvable</u> if Γ has a <u>solvable subgroup</u> of <u>finite index</u>.

In the next lecture we will see that a virtually solvable group does not have a non-commutative free subgroup.