Lecture 23: Virtually solvable group does not have a non-commutative free subgroup

Wednesday, November 22, 2017 10:48 AM

In the previous lecture we stated Tits alternative:

A f.g. linear group either is virtually solvable or it has a non-commutative free subgp.

There are many ways to show that a virtually solvable gp does not contain a non-commutative free subgp. We use finite gps to prove this statement.

<u>Proposition</u>. A virtually solvable gp Γ does NOT have a non-commutative free subgp.

<u>Pf.</u> Suppose to the contrary that $\exists\, a,b \in F$ which freely generate a subgp F. And suppose $[\Gamma:\Lambda]<\infty$ and Λ is solvable. Let $N := \mathrm{core}(\Lambda)$.

As we have seen before $[\Gamma:N] \le |S_{\Gamma/\Lambda}| < \infty$, and N is solvable.

Hence $F/_{F\cap N} \simeq FN/_N \le \Gamma/_N$ implies $[F:F\cap N] = m < \infty$.

Let $n \in \mathbb{Z}^+$ be s.t. $n! > 2m$ and $n \ge 5$. Since $S_n = \langle (1\,2), (1 \cdots n)\rangle$, there is an onto gp homo. $\phi: F \twoheadrightarrow S_n$. So $\phi(F\cap N) \triangleleft S_n$. Hence $\phi(F\cap N) = 1$, A_n, or S_n. Since $[S_n : \phi(F\cap N)] \le [F : F\cap N] = m$ and

Lecture 23: Virtually solvable groups cannot have free subgroups

Tuesday, November 21, 2017 10:43 PM

$2m < n!$, we deduce $\phi(F\cap N) \supseteq A_n$. As $F\cap N$ is solvable, we get that A_n should be solvable. But this is a contradiction, as A_n is a non-commutative simple group for $n \geq 5$. ■

In this proof, you can see an important idea: sometimes we use certain finite gps as "observers" to understand properties infinite gps.

<u>Def.</u> Suppose $R \subseteq F(X)$. Then $\langle X \mid R\rangle$ means the group $F(X)/N$ where N is smallest normal subgp of $F(X)$ that contains R; that means

$$N = \langle \bigcup_{g\in F(X)} gRg^{-1} \rangle.$$

<u>Ex.</u> $\mathbb{Z}/n\mathbb{Z} \simeq \langle a \mid a^n \rangle$

<u>Pf</u> $\exists\, \tilde{\theta}: \langle \tilde{a}\rangle \to \mathbb{Z}/n\mathbb{Z}$, $\tilde{\theta}(\tilde{a}) = 1 + n\mathbb{Z}$;

$\tilde{a}^n \in \ker\tilde{\theta} \Rightarrow \exists\, \theta: \langle a \mid a^n\rangle \to \mathbb{Z}/n\mathbb{Z}$,

$\theta(a) = 1 + n\mathbb{Z}$.

For any $m \in \mathbb{Z}$, $a^m = a^{n\lfloor m/n\rfloor + (m - \lfloor m/n\rfloor n)} = a^{m-\lfloor m/n\rfloor n} \in \{e, a, \ldots, a^{n-1}\}$.

So $|\langle a \mid a^n\rangle| \leq n$. Since θ is onto and $|\langle a \mid a^n\rangle| \leq n$, we have that θ is an isomorphism. ■

Lecture 23: Presentations

Monday, November 27, 2017 10:21 AM

To show $\langle X \mid R\rangle$ is isomorphic to a given finite gp G, a general approach is as follows:

easy and general steps

<u>Step 1.</u> Find a generating set $\overline{X}$ of G which satisfies the given relations R. needs care

<u>Step 2.</u> Use the universal property of the free gp $\langle X\rangle$ to define a gp hom. $\widetilde{\Phi}:\langle X\rangle \to G$, $\widetilde{\Phi}(x) := \overline{x}$.

<u>Step 3.</u> Check the relations; this shows $R \subseteq \ker \widetilde{\Phi}$. So the smallest normal subgp N that contains R as a subset is a subgp of $\ker\widetilde{\Phi}$. So

$$\exists \text{ a gp hom. } \Phi:\langle X \mid R\rangle \to G,\ \Phi(x) := \overline{x}.$$

Since $\overline{X} \subseteq \operatorname{Im}\Phi$, $\langle \overline{X}\rangle \subseteq \operatorname{Im}\Phi$. And so Φ is onto.

<u>Step 4.</u> Use the relations to show The most tricky part

$$|\langle X \mid R\rangle| \le |G|.$$

<u>Step 5.</u> Deduce that $\Phi:\langle X\mid R\rangle \to G$ is an isomorphism.

Lecture 23: Presentation of the dihedral group

Monday, November 27, 2017 10:31 AM

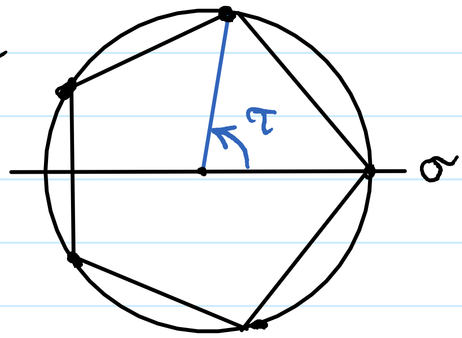
<u>Ex</u>. Prove that $\langle a,b \mid a^2, b^n, \underset{\sim}{aba^{-1}=b^{-1}} \rangle \simeq D_{2n}$

[we sometimes write $\omega_1=\omega_2$ instead of writing $\omega_1^{-1}\omega_2$.]

<u>Pf</u>. We know that $D_{2n}=\{id., \tau, \ldots, \tau^{n-1}, \sigma, \sigma\tau, \ldots, \sigma\tau^{n-1}\}$

where σ is the reflection about the x-axis,

and τ is the rotation by $\frac{2\pi}{n}$ about the origin. So



$$\tau(z)=e^{2\pi i/n}z \quad \text{and} \quad \sigma(z)=\bar{z}.$$

So $\sigma^2(z)=\bar{\bar{z}}=z$; $\tau^n(z)=\left(e^{2\pi i/n}\right)^n z=z$; and

$$\sigma\tau\sigma^{-1}(z)=\sigma\tau(\bar{z})=\sigma(e^{2\pi i/n}\bar{z})=e^{-2\pi i/n}\cdot z$$
$$=\tau^{-1}(z).$$

Hence the onto group hom $\phi:\langle a,b\rangle \to D_{2n}$ factors through

$\Phi:\langle a,b \mid a^2, b^n, aba^{-1}=b^{-1}\rangle \longrightarrow D_{2n}$.

<u>Claim</u>. $\{1, \bar{b}, \ldots, \bar{b}^{n-1}, \bar{a}, \bar{a}\bar{b}, \ldots, \bar{a}\bar{b}^{n-1}\}$ is a group.

<u>Pf</u>. Since it is finite, it is enough to check that it is closed under multiplication. Since $\bar{b}^n=1$, $\{1,\bar{b},\ldots,\bar{b}^{n-1}\}$ is a

Lecture 23: Presentation of the dihedral group

Tuesday, November 21, 2017 11:12 PM

gp. $\bar{a}\bar{b}^i \, \bar{b}^j$ ✓

$$\bar{b}^j \bar{a}\bar{b}^i = \bar{a}\,\bar{a}\,\bar{b}^j\,\bar{a}\,\bar{b}^i = \bar{a}\cdot(\bar{a}\,\bar{b}\,\bar{a}^{-1})^j\,\bar{b}^i$$

$$= \bar{a}\cdot\bar{b}^{-j}\,\bar{b}^i \quad ✓$$

$$\bar{a}\bar{b}^j\,\bar{a}\bar{b}^i = \bar{b}^{-j}\cdot\bar{b}^i \quad ✓$$

So $|\langle a,b \mid a^2, b^n, aba^{-1}=b^{-1}\rangle| \le 2n$.

Since $\phi: \langle a,b \mid a^2, b^n, aba^{-1}=b^{-1}\rangle \to D_{2n}$ is <u>onto</u> and $|D_{2n}| = 2n$, we deduce that ϕ is an isomorphism. ■

In the HW assignment, using universal properties of the free gp and the free product of gps, you will prove that

$$\langle X_1 \mid R_1\rangle * \langle X_2 \mid R_2\rangle \simeq \langle X_1 \sqcup X_2 \mid R_1 \sqcup R_2\rangle.$$

<u>Ex.</u> Prove that $\left\langle \overline{\begin{bmatrix}1 & 2\\ & 1\end{bmatrix}}, \overline{\begin{bmatrix} & 1\\ -1 & \end{bmatrix}}\right\rangle \simeq \langle a,b \mid b^2\rangle$ where

$\bar{g} := g\{\pm I\} \in PSL_2(\mathbb{R})$.

<u>Pf.</u> We have proved that $\left\langle \overline{\begin{bmatrix}1 & 2\\ & 1\end{bmatrix}}, \overline{\begin{bmatrix} & 1\\ -1 & \end{bmatrix}}\right\rangle \simeq \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$

$\simeq \langle a\rangle * \langle b \mid b^2\rangle \simeq \langle a,b \mid b^2\rangle$.

■