Homework 1

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1. For a group G, let $Aut(G) := \{ f : G \to G \mid f \text{ is an automorphism} \}$.

We know that $(Aut(G), \circ)$ is a group. Let $c : G \to Aut(G)$,

$c(g) := c_g$ where $c_g(g') := g g' g^{-1}$.

(a) Prove that $c_g \in Aut(G)$ and c is a group homomorphism.

(b) Image of c is called the group of inner automorphisms, and it is denoted by $Inn(G)$. Prove that $\ker c = Z(G)$ and deduce $Inn(G) \simeq G/Z(G)$.

(c) Prove that $Inn(G) \triangleleft Aut(G)$.

(d) Prove that $|Z(Aut(G))| \le |Hom(G, Z(G))|$; in particular, if either $Z(G) = 1$ or G is perfect (that means $G = [G,G]$), then $Z(Aut(G))$ is trivial.

(Hint ① $\forall g \in G$ and $\forall \phi \in Aut(G)$, $\phi \circ c_g \circ \phi^{-1} = c_{\phi(g)}$;

② If $\phi \in Z(Aut(G))$, then $c_g = c_{\phi(g)}$; and so $\phi(g) = g\, \eta(g)$ for some $\eta(g) \in Z(G)$.

③ Prove $\eta \in Hom(G, Z(G))$.)

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2. Let $SL_2(\mathbb{R})$ be the set real 2×2 matrices with determinant 1.

For $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathbb{C} \cup \{\infty\}$

$$(\circledast) \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \begin{cases} \dfrac{az+b}{cz+d} & \text{if } z \neq \infty, cz+d \neq 0. \\ \infty & \text{if } z \neq \infty, cz+d=0, \\ \dfrac{a}{c} & \text{if } z = \infty. \end{cases}$$

ⓐ Prove that $\circledast$ defines a group action $SL_2(\mathbb{R}) \curvearrowright \mathbb{C} \cup \{\infty\}$.

ⓑ Convince yourself that $\operatorname{Im}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z\right) = \dfrac{\operatorname{Im}(z)}{|cz+d|^2}$, where $\operatorname{Im}(z)$ is the imaginary part of z. Prove that $SL_2(\mathbb{R})$ has three orbits:

the upper half plane $\mathcal{H}$, the real axis $\cup \{\infty\}$, and the lower half plane $\mathcal{H}^-$.

ⓒ Show that the stabilizer of i is the special orthogonal group $SO_2(\mathbb{R}) := \{ g \in SL_2(\mathbb{R}) \mid g g^t = I \}$.

3. Show that $\Psi : Act(G,X) \longrightarrow Hom(G, S_X)$,

$$((\Psi(m))(g))(x) := m(g,x)$$

and

$$\Phi : Hom(G, S_X) \longrightarrow Act(G,X),$$

$$(\Phi(f))(g,x) := (f(g))(x)$$

are inverse of each other.

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4. Suppose G is a finite group, $C \subseteq \mathbb{R}^n$ is a convex subset; that means, if $p, q \in C$, then the segment pq is in C. Suppose $G \curvearrowright C$ by affine actions; that means $\forall p, q \in C$, $\forall t \in [0,1]$, $\forall g \in G$,

$$g\cdot(tp+(1-t)q) = t\, g\cdot p + (1-t)\, g\cdot q.$$

Prove that G has a fixed point; that means

$$\exists\, x \in C \text{ s.t. } \forall g \in G,\ g\cdot x = x.$$

(Hint. ① Suppose $c_1, \ldots, c_n \in C$. By the convexity of C, using induction show the average $\frac{1}{n}(c_1 + c_2 + \cdots + c_n)$ is in C.

② Take $y \in C$, and let x be the average of the G-orbit of y. Prove that x is a fixed point of G.)

5. Suppose G is a finite subgroup of the group $GL_n(\mathbb{R})$ of $n \times n$ real invertible matrices. Prove that there is an inner product on $\mathbb{R}^n$ which is G-invariant.

(Recall. $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called an inner product if

ⓐ $\langle c_1 v_1 + c_2 v_2, w\rangle = c_1\langle v_1, w\rangle + c_2\langle v_2, w\rangle$

ⓑ $\langle v, c_1 w_1 + c_2 w_2\rangle = c_1\langle v, w_1\rangle + c_2\langle v, w_2\rangle$

ⓒ $\langle v, v\rangle > 0$ if $v \neq 0$.

ⓓ $\langle v, w\rangle = \langle w, v\rangle$

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for instance $(a_1,\dots,a_n)\bullet(b_1,\dots,b_n) = a_1b_1+a_2b_2+\dots+a_nb_n$

is an inner product.)

(Hint. Define $\langle v,w\rangle := \frac{1}{|G|}\sum_{g\in G} gv\bullet gw$

(the average of the standard inner product along the

G-orbits of v and w.); you have to show $\langle\,,\,\rangle$ is

an inner product and $\langle gv,gw\rangle = \langle v,w\rangle$.)

[This problem is extremely useful as it implies:

if $V\subseteq\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ which is invariant

under G (that means $\forall v\in V, \forall g\in G$, we have $g\cdot v\in V$.)

then $V^\perp := \{w\in\mathbb{R}^n \mid \forall v\in V, \langle w,v\rangle = 0\}$ is

also G-invariant, and $V\oplus V^\perp = \mathbb{R}^n$.]

6. Recall that we say $G\curvearrowright X$ transitively if $|_G\backslash^X| = 1$.

A transitive group action $G\curvearrowright X$ is called <u>primitive</u> if

it does not preserve any non-trivial partition of X, where

trivial partitions are $\{X\}$ and $\{\{x\}\mid x\in X\}$.

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For instance, let $\sigma : \{1,2,3,4\} \to \{1,2,3,4\}$,

$1 \overset{\sigma}{\longmapsto} 2 \overset{\sigma}{\longmapsto} 3 \overset{\sigma}{\longmapsto} 4 \overset{\sigma}{\longmapsto} 1$. Then

$\{\{1,3\},\{2,4\}\}$ is

preserved by $\langle\sigma\rangle$; so $\langle\sigma\rangle \curvearrowright \{1,2,3,4\}$ is NOT primitive

though it is transitive.

Suppose $G \curvearrowright X$ is a non-trivial transitive Then

$G \curvearrowright X$ is primitive if and only if for any $x \in X$

the stabilizer group G_x of x is a <u>maximal</u> subgroup;

that means ① G_x is a proper subgp

② $G_x \le H \le G \Rightarrow$ either $G_x = H$ or $G = H$.

(<u>Hint</u>. Since $G \curvearrowright X$ is transitive, $X = G \cdot x$;

If $\exists\, G_x \lneq H \lneq G$, then show that $\{gH \cdot x \mid g \in G\}$

is a non-trivial partition of X which is preserved by $G \curvearrowright X$.

- Suppose $\{X_i \mid i \in I\}$ is a partition which is preserved

by the G-action. So $\forall g$, $g \cdot X_i = X_{\sigma_g(i)}$ where $\sigma_g \in S_I$.

Suppose $|X_0| \ge 2$; and $x \in X_0$.

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$\forall g \in G_x$, $g \cdot X_0 \cap X_0 \neq \emptyset$, which implies $gX_0 = X_0$.

So $G_{X_0} \supseteq G_x$. Since $|X_0| \geq 2$ and $G \curvearrowright X$ is transitive,

$G_{X_0} \supsetneq G_x$. Since $\exists x' \in X \setminus X_0$ and $G \curvearrowright X$ is trans. $G_{X_0} \subsetneq G$.)

7. Suppose $G \curvearrowright X$ transitively. Prove that the kernel of this group action is the normal core $\mathrm{cor}(G_x)$ of the stabilizer group of a point $x \in X$.