Homework 2

Thursday, October 11, 2018 7:50 PM

1. Suppose p is an odd prime, G is a group of order $p(p+1)$ that does not a normal subgroup of order p. Prove that p is a Mersenne prime; that means $p = 2^n - 1$ for some $n \in \mathbb{Z}^+$.

(Hint. Go through the proof given in lecture.)

2. Suppose $p < q < \ell$ are three primes, and G is a group of order $pq\ell$. Prove that G has a normal subgp of order ℓ.

(Hint. First prove G has a normal subgroup of order either p, q, or ℓ.)

3. Suppose G is a finite group, and $N \triangleleft G$. Let $P \in \mathrm{Syl}_p(N)$. Prove that $G = N_G(P)\, N$.

(Hint. $G \curvearrowright \mathrm{Syl}_p(N)$ by conjugation + 2nd Sylow theorem)

4. Suppose G is a finite group, $N \trianglelefteq G$, and p is a prime factor of $|N|$.

(a) Suppose $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_p(N)$. Prove that $\exists g \in G$ s.t. $Q = gPg^{-1} \cap N$.

Homework 2

Thursday, October 11, 2018 10:37 PM

(b) Prove that the following is a well-defined surjective function

$$\mathrm{Syl}_p(G) \xrightarrow{\Phi} \mathrm{Syl}_p(N),$$
$$P \longmapsto P \cap N.$$

(c) For $P \in \mathrm{Syl}_p(G)$, let $Q := \Phi(P)$. Prove that

$$N_G(P) \subseteq N_G(Q) \quad \text{and} \quad |\Phi^{-1}(Q)| = [N_G(Q) : N_G(P)].$$

(d) Prove that $|\mathrm{Syl}_p(N)| \,\big|\, |\mathrm{Syl}_p(G)|$.

5. A group is called <u>simple</u> if it does not have a non-trivial proper normal subgroup; that means $N \trianglelefteq G \Rightarrow N = \{e\}$ or $N = G$. Prove that a group of order p^2q is not simple.

6. Prove that a group of order 36 is not simple.

(<u>Hint</u>: Suppose G is simple; find $|\mathrm{Syl}_3(G)|$; consider $G \curvearrowright \mathrm{Syl}_3(G)$ by conjugation and show it has a non-trivial kernel.)

7. Let G be a finite group and $H \leq G$. Suppose $\forall x \in H \setminus \{1\}$, $C_G(x) \subseteq H$. Prove that $\gcd(|H|, [G:H]) = 1$.

Homework 2

Thursday, October 11, 2018 11:12 PM

(Hint. Suppose $p \mid \gcd(|H|, [G:H])$; consider $Q \in \mathrm{Syl}_p(H)$; find $P \in \mathrm{Syl}_p(G)$ s.t. $Q \subseteq P$. Show $Z(P) \subseteq Q$.)