Homework 3

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1. Prove that $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times$, where

$$(\mathbb{Z}/n\mathbb{Z})^\times := \{ a+n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z} \mid \exists a',\ \underbrace{(a+n\mathbb{Z})(a'+n\mathbb{Z})}_{=1+n\mathbb{Z}} \}.$$

2. Suppose $f_1, f_2 \in \mathrm{Hom}(H, \mathrm{Aut}(N))$. Suppose there is an isomorphism $\Phi: H \ltimes_{f_1} N \to H \ltimes_{f_2} N$ such that $\circledast$ is a commuting diagram.

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] \arrow[d, "="] & H \ltimes_{f_1} N \arrow[r] \arrow[d, "\Phi"] & H \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & N \arrow[r] & H \ltimes_{f_2} N \arrow[r] & H \arrow[r] & 1 \end{tikzcd} \qquad \circledast$$

Let $\sigma: H \to \mathrm{Aut}(N)$ be the following function: $\sigma(h) = f_2(h) \circ f_1(h)^{-1}$.

(a) Prove that $\sigma(h) \in \mathrm{Inn}(N)$ for any $h \in H$.

(b) Prove that, $\forall\, h_1, h_2 \in H$,

$$\sigma(h_1 h_2) = \sigma(h_1) \circ f_1(h_1) \circ \sigma(h_2) \circ f_1(h_1)^{-1}$$

(1-cocycle relation)

(Hint (a) Show $\Phi(h,1) = (h, n(h))$; Consider $\Phi((h,1)(1,n)(h,1)^{-1})$.

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3. Suppose $N_1, \dots, N_k$ are normal subgroups of G, and, for any i, $N_i \cap N_1 \cdots N_{i-1} N_{i+1} \cdots N_k = \{1\}$. Prove that

$$N_1 \times N_2 \times \cdots \times N_k \longrightarrow N_1 \cdot N_2 \cdot \cdots \cdot N_k$$
$$(x_1, x_2, \cdots, x_k) \longmapsto x_1 \cdot x_2 \cdot \cdots \cdot x_k$$

is a group isomorphism.

4. Suppose in a finite group G the following property holds:

$(*)$ $H \lneq G \Rightarrow H \lneq N_G(H)$.

ⓐ Prove that all the Sylow subgroups are normal.

Deduce that $\forall p \mid |G|$, there is a unique Sylow p-subgp P_p.

ⓑ Prove that $G \simeq \prod_{p \mid |G|} P_p$.

(Hint: If $P \in Syl_p(G)$, what do we know about $N_G(N_G(P))$?

• Use previous problem.)

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5. Suppose A is an abelian normal subgroup of G. Let $H := G/A$.

Suppose $G = \bigsqcup_{i=1}^{m} g_i A$ (so $|H| = m$). Let $h_i := g_i A \in H$.

For any i, j, $g_i A \cdot g_j A = g_{k(i,j)} A$ for some $k(i,j)$

$\Rightarrow g_k^{-1} g_i g_j \in A$. So we get a function

$$c : H \times H \longrightarrow A, \quad c(h_i, h_j) := g_{k(i,j)}^{-1} g_i g_j .$$

(c depends on the choice of representatives g_i's; think about g_i's as a section; that means $s : H \to G$, $s(h_i) = g_i$; and notice $s(h) A = h$.)

Notice that $G \curvearrowright A$ by conjugation, and, since A is abelian, A is in the kernel of this action; this implies $H = G/A$ acts on A. For $h = gA$ and $a \in A$, we let ${}^{h}a := g a g^{-1}$.
(so ${}^{h_1}({}^{h_2}a) = {}^{h_1 h_2}a$ as it is an action; and ${}^{h}(a_1 a_2) = {}^{h}a_1 \, {}^{h}a_2$.)

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(a) Prove that for any $h_1, h_2, h_3 \in H$ we have

$$c(h_1h_2, h_3)\ {}^{h_3^{-1}}c(h_1,h_2) = c(h_1, h_2h_3)\ c(h_2,h_3).$$

[It is better to use the section $s: H \to G$, $s(h_i) = g_i$; then

$s(h_1)\, s(h_2) = s(h_1h_2)\, c(h_1, h_2)$.

$(s(h_1)\, s(h_2))\, s(h_3) = s(h_1)\,(s(h_2)\, s(h_3))$

$\overset{?}{\Rightarrow}$ $s(h_1h_2)\, c(h_1,h_2)\, s(h_3) = s(h_1)\, s(h_2h_3)\, c(h_2,h_3)$

$\overset{?}{\Rightarrow}$ $s(h_1h_2)\, s(h_3)\ {}^{h_3^{-1}}c(h_1,h_2) = s(h_1h_2h_3)\, c(h_1, h_2h_3)\, c(h_2,h_3)$

$\overset{?}{\Rightarrow}$ $s(h_1h_2h_3)\, c(h_1h_2, h_3)\ {}^{h_3^{-1}}c(h_1,h_2) = s(h_1h_2h_3)\ c(h_1, h_2h_3)\ c(h_2,h_3)$.]

(This is called the 2-cocycle relation.)

(b) Prove that the short exact sequence $1 \to A \to G \to H \to 1$

splits if and only if $\exists$ a function $\alpha: H \to A$ such that

$$c(h_1,h_2) = {}^{h_2^{-1}}\alpha(h_1)\,\alpha(h_2)\,\alpha(h_1h_2)^{-1}$$

[Hint. ($\Rightarrow$) Suppose $\psi: H \to G$ is the splitting homomorphism. Then

$\forall h \in H$, $\psi(h)A = s(h)A$. Let $\alpha: H \to A$, $\alpha(h) := \psi(h)^{-1} s(h)$.

Use $s(h_1)\, s(h_2) = s(h_1h_2)\, c(h_1,h_2)$ to check the relation.

($\Leftarrow$) Let $\psi(h) := s(h)\,\alpha(h)^{-1}$. Use the given relation to show $\psi: H \to G$

is a group hom. And notice $\psi(h)A = h$.]

(This is called the 2-boundary relation.)

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(c) Suppose $\gcd(|A|,|H|)=1$. Prove that a 2-cocyle $c: H\times H \to A$ is a 2-boundary; and so $1\to A\to G\to H\to 1$ splits. (The abelian case of the Schur-Zassenhaus theorem.)

[Hint: The trick is "taking average"; in this part, proof would be more clear if we use $+$ for the operation in A (notice that A is abelian). So c satisfies:

$$c(h_1h_2,h_3) + {}^{h_3^{-1}}c(h_1,h_2) = c(h_1,h_2h_3) + c(h_2,h_3). \quad (*)$$

Let $\alpha(h) := \frac{1}{|H|}\sum_{h_1\in H} c(h_1,h)$ (why does it make sense? Here is where we are using $\gcd(|A|,|H|)=1$.)

$(*)$ implies (why?)

$$\underbrace{\frac{1}{|H|}\sum_{h_1\in H} c(h_1h_2,h_3)}_{\alpha(h_3)} + \underbrace{{}^{h_3^{-1}}\left(\frac{1}{|H|}\sum_{h_1\in H} c(h_1,h_2)\right)}_{{}^{h_3^{-1}}\alpha(h_2)} = \underbrace{\frac{1}{|H|}\sum_{h_1\in H} c(h_1,h_2h_3)}_{\alpha(h_2h_3)} + c(h_2,h_3)$$

$$\alpha(h_3) + {}^{h_3^{-1}}\alpha(h_2) = \alpha(h_2h_3) + c(h_2,h_3)$$

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$H \leq G$ is called a <u>characteristic subgroup</u> if for any $\theta \in \mathrm{Aut}(G)$ $\theta(H) = H$. Convince yourself that any characteristic subgroup is a normal subgroup.

6. (a) Suppose $N \triangleleft G$ and K is a characteristic subgroup of N. Prove that $K \triangleleft G$.

(b) We say a group H is <u>characteristically simple</u> if its only char. subgroups are $\{1\}$ and H.

Suppose N is a <u>minimal</u> normal subgroup of G; that means, if $K \triangleleft G$ and $K \lneqq N$, then $K = \{1\}$, and $N \neq \{1\}$.

Prove that N is characteristically simple.

7. (a) Suppose P is a Sylow p-subgp of G, and $P \triangleleft G$. Prove that P is a characteristic subgroup of G.

(b) Suppose $H \triangleleft G$ and $\gcd(|H|, [G:H]) = 1$. Prove that H is a characteristic subgroup of G.