Homework 4

Friday, October 26, 2018 1:37 AM

$\boxed{1}$ In this problem, you prove that $Aut(S_n) = Inn(S_n)$ if $n \geq 7$. (All the automorphisms of S_n are inner.) Suppose $\varphi \in Aut(S_n)$.

(a) Suppose $n \geq 5$, and φ sends transpositions to transpositions; that means $|\mathrm{Supp}(\varphi(a\ b))| = 2$ for any $1 \leq a < b \leq n$. Prove that φ is an inner automorphism.

[Hint ① Suppose τ_1 and τ_2 are two transpositions. Observe:

τ_1 and τ_2 do not commute if and only if $|\mathrm{supp}(\tau_1) \cap \mathrm{supp}(\tau_2)| = 1$.

② Any transposition gives us an edge in the complete graph with n vertices; by assumption φ induces a bijection on the edges of the complete graph. ① implies two edges with a common vertex are mapped to two edges with a common vertex. Use this to get a permutation σ on vertices.

③ Show that for any transposition τ, $\sigma\, \varphi(\tau)\, \sigma^{-1} = \tau$.]

(b) Prove that $\varphi(\sigma_1)$ and $\varphi(\sigma_2)$ are conjugate if and only if σ_1 and σ_2 are conjugate.

(c) Let T_k be the set of permutations with cycle type $\underbrace{2,\ldots,2}_{k},\underbrace{1,\ldots,1}_{n-2k}$; for instance T_1 consists of transpositions. Show that

Homework 4

Friday, October 26, 2018 2:14 PM

$$|T_k| = n(n-1)\cdots(n-2k+1)/_{k!\,2^k} \geq \frac{n(n-1)}{2} \cdot \frac{(2k-2)!}{k!\cdot 2^{k-1}}.$$

(d) Prove that $\varphi(T_1) = T_k$ for some $1 \leq k \leq n/2$. (Use part (b))

(e) Prove that $\varphi(T_1) = T_1$; and deduce that $\varphi \in \mathrm{Inn}(S_n)$.

$\boxed{2}$. In this problem, you prove that $\mathrm{Aut}(S_6) \neq \mathrm{Inn}(S_6)$.

(In this problem you can use the fact that A_n is simple if $n \geq 5$)

(a) Show that S_5 has 6 Sylow 5-subgroups. Deduce that S_6 has a subgroup H which is isomorphic to S_5 and acts transitively on $\{1,2,\dots,6\}$. And so $\mathrm{Fix}(\sigma H \sigma^{-1}) = \emptyset$ for any $\sigma \in S_6$.

(b) Consider $S_6 \curvearrowright S_6/H$ by the left translations. Since $|H| = |S_5|$, we have $|S_6/H| = 6$. So the above action gives us a group homomorphism $\varphi : S_6 \to S_6$. Prove that φ is an isomorphism.

(c) Show that $\mathrm{Fix}(\varphi(H)) \neq \emptyset$, and deduce φ is NOT inner automorphism of S_6.

Homework 4

Friday, October 26, 2018 4:16 PM

. One of the important result in finite group theory is the following result of Burnside:

Burnside's normal p-complement theorem.

Suppose G is a finite group, $1 \neq P$ is a Sylow p-subgroup, and $P \subseteq Z(N_G(P))$. Then $\exists\ N \triangleleft G$ s.t. $|N| = |G/P|$.

This is an extremely useful theorem; for instance try to use this to give a short of a result we have proved earlier: a group G of order $p(p+1)$ has a normal subgroup of order p or $p+1$. (This is not part of the problem). In this problem you will see the powerful combination of this theorem with the Schur-Zassenhaus theorem:

$\boxed{3}$ Suppose $\gcd(n, \varphi(n)) = 1$, and G is a group of order n. Prove that a group of order n is cyclic.

(Hint: Arith. observations: . $\gcd(n, \varphi(n)) = 1 \Rightarrow n$ is square-free

. $\left.\begin{array}{l}\gcd(n, \varphi(n)) = 1 \\ m \mid n\end{array}\right\} \Rightarrow \gcd(m, \varphi(n)) = \gcd(m, \varphi(m)) = \gcd(n, \varphi(m)) = 1.$

. Use strong induction on n; and the mentioned theorems.)

Homework 4

Friday, October 26, 2018 4:22 PM

$\boxed{4}$ Suppose G is a finite group and for any $d \in \mathbb{Z}^+$, $|\{g \in G \mid g^d = e\}| \leq d$. Prove that G is cyclic.

(Hint: Let $X_d := \{g \in G \mid o(g) = d\}$ and $\psi(d) := |X_d|$.

Step 1. Show, if $\psi(d) \neq 0$, then $\psi(d) = \phi(d)$.

Step 2. Notice $\sum_{d|n} \psi(d) = n$ where $n = |G|$.

Step 3. From arithmetic we know $\sum_{d|n} \phi(d) = n$. (you are allowed to use this without proof.) Use steps 1 and 2 to show $d|n \Rightarrow \psi(d) = \phi(d)$; and finish proof.)

$\boxed{5}$ For a group G, let $[G,G]$ be the subgroup generated by $[g_1, g_2] := g_1^{-1} g_2^{-1} g_1 g_2$ for $g_1, g_2 \in G$.

(a) Show that $[G,G]$ is a characteristic subgroup of G.

(b) For $N \lhd G$, prove that G/N is abelian if and only if $[G,G] \subseteq N$.

(c) Prove that $[S_n, S_n] = A_n$ if $n \geq 3$.

Homework 4

Friday, October 26, 2018 4:33 PM

$\boxed{6}$ Prove that there is no finite group G such that

$$[G,G] \simeq S_4 .$$

(Hint. Suppose to the contrary that there is such a group G. Convince yourself that $P := \{I, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ is the unique Sylow 2-subgp of A_4; and so P is a characteristic subgroup of A_4.

- G acts by conjugation on $[G,G] \simeq S_4$; argue why this induces an action on $A_4/P \simeq \mathbb{Z}/3\mathbb{Z}$;
- Argue why $[G,G]$ should act trivially on A_4/P; and deduce $S_4 \curvearrowright A_4/P$ by conjugation should be the trivial action.
- Check that $(1\ 2)(1\ 2\ 3)(1\ 2)P \neq (1\ 2\ 3)P$, and get a contradiction.)

$\boxed{7}$ (a) Prove that $\langle (1\ 2), (1\ 2 \cdots n) \rangle = S_n$.

(b) Suppose p is an odd prime, $\tau \in S_p$ is a transposition and $\sigma \in S_p$ has order p. Prove that $\langle \tau, \sigma \rangle = S_p$.

Homework 4

Friday, October 26, 2018 5:06 PM

(<u>Hint</u>. (a) Let $H := \langle (1\ 2), (1\ 2 \cdots n) \rangle$. Notice $(1\ 2)(2\ 3 \cdots n) = (1\ 2 \cdots n)$ and so $\gamma := (2\ 3 \cdots n) \in H \Rightarrow \gamma^{i} (1\ 2) \gamma^{-i} \in H \Rightarrow (1\ j) \in H \ \forall j \Rightarrow$ $(1\ i)(1\ j)(1\ i) = (i\ j) \in H$.

(b) After reordering, we can assume $\sigma = (1\ 2 \cdots p)$. Let $H = \langle (a\ b), (1\ 2 \cdots p) \rangle$; argue why we can further assume $H = \langle (1\ b), \underbrace{(1\ 2 \cdots p)}_{\sigma} \rangle$ after another reordering if needed; using σ^{i} and part (a) finish the proof.)