Homework 6

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1. (a) Suppose V is a vector space and G is a perfect group subgroup of $GL(V)$; i.e. $G=[G,G]$. Suppose no non-trivial subspace of V is invariant under G; that means if W is a subspace of V and, for any $g\in G$, $gW=W$, then $W=0$ or $W=V$.

Prove that $H:=G\ltimes V$ is perfect.

(b) Let $V:=\mathbb{R}^{2n}$ and $f: V\times V\longrightarrow \mathbb{R}$,

$$f((x_1,\dots,x_{2n}),(y_1,\dots,y_{2n})):=x_{n+1}y_1+\cdots+x_{2n}y_n-x_1y_{n+1}-\cdots-x_ny_{2n}$$

Convince yourself that f is a non-degenerate bilinear form; that means:

$$f(c_1v_1+c_2v_2,\omega)=c_1f(v_1,\omega)+c_2f(v_2,\omega)$$
$$f(v,c_1\omega_1+c_2\omega_2)=c_1f(v,\omega_1)+c_2f(v,\omega_2)$$
$$f(v,V)=0 \Rightarrow v=0$$
$$f(V,\omega)=0 \Rightarrow \omega=0$$

And it is symplectic; that means $f(v,v)=0$ for any $v\in V$.

Let $H(V,f):=\{(v,c)\mid v\in\mathbb{R}^{2n}, c\in\mathbb{R}\}$, and

$$(v_1,c_1)\cdot(v_2,c_2):=(v_1+v_2,\ c_1+c_2+\tfrac{1}{2}f(v_1,v_2)).$$

Convince yourself that

$$H(V,f)\longrightarrow\left\{\begin{bmatrix}1&\omega^t&c\\&I&\omega'\\&&1\end{bmatrix}\ \middle|\ \omega,\omega'\in\mathbb{R}^n,\ c\in\mathbb{R}\right\}$$

$$((\omega,\omega'),c)\longmapsto\begin{bmatrix}1&\omega^t&c+\frac{1}{2}\omega^t\omega'\\&I&\omega'\\&&1\end{bmatrix}$$

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is a group isomorphism. $H(V,f)$ is called the Heisenberg group.

(b-1) Prove that $[(v_1,c_1),(v_2,c_2)] = -f(v_1,v_2)$.

(b-2) Prove that $Z(H(V,f)) = 0 \oplus \mathbb{R}$ and

$$0 \to \mathbb{R} \to H(V,f) \to V \to 0$$

$$c \mapsto (0, c)$$

$$(v, c) \mapsto v$$

is a short exact sequence.

(b-3) Prove that $[H(V,f), H(V,f)] = 0 \oplus \mathbb{R}$, and $H(V,f)$ is nilpotent.

(c) Let $Sp_{2n}(\mathbb{R}) := \{ g \in GL(V) \mid f(g.v, g.w) = f(v,w)$ for any $v,w \in V \}$ (it is called symplectic group.) For $g \in Sp_{2n}(\mathbb{R})$, let $g\cdot(v,c) := (g.v, c)$. Prove that this map embeds $Sp_{2n}(\mathbb{R})$ into $Aut(H(V,f))$.

(d) We know that $Sp_{2n}(\mathbb{R})$ is perfect and no non-trivial subspace of $\mathbb{R}^{2n}$ is invariant under $Sp_{2n}(\mathbb{R})$. Prove that $Sp_{2n}(\mathbb{R}) \ltimes H(V,f)$ is perfect.

(e) Prove that $(0 \oplus \mathbb{R}) \subseteq Z(Sp_{2n}(\mathbb{R}) \ltimes H(V,f))$.

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2. Suppose G is a group of order $2^k m$ where m is odd.

Suppose a Sylow 2-subgroup P of G is cyclic. Prove that

G has a characteristic subgroup of order m.

[Hint. Point 1. show that $G \xrightarrow{\Phi} S_G \xrightarrow{\epsilon} \{\pm 1\}$ is non-trivial;

similar to what we did in class.

Point 2. Show that $\ker \epsilon \circ \Phi$ is a characteristic

subgp of index 2.

Point 3. Use induction and deduce that there are

char. subgps of order $2^i m$ for any $0 \le i \le k$.]

(Please do not use Burnside's normal p-compl. theorem)

3. Suppose G is a finite group and $H \le G$.

(a) Prove that $H = G \iff H\Phi(G) = G$.

(b) Let $\pi : G \to G/\Phi(G)$ be the natural projection map. Suppose

$S \subseteq G$. Prove that $\langle S \rangle = G \iff \langle \pi(S) \rangle = G/\Phi(G)$.

In particular $\langle S \rangle = G \iff \langle S \setminus \Phi(G) \rangle = G$.

(c) Let $d(G) :=$ the min. number of generators of G.

Prove that $d(G) = d(G/\Phi(G))$.

[Hint (a) If $H \ne G$, then $\exists$ a max. subgp $H \le M < G$.

$\Rightarrow M/\Phi(G) \ne G/\Phi(G)$.]

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4. Suppose G is a finite group; and $\Phi(G)$ is the Frattini subgroup of G.

(a) Suppose P is a Sylow subgroup of $\Phi(G)$. Prove that

$$P \trianglelefteq G.$$

(b) Prove that $\Phi(G)$ is nilpotent.

[Hint. (a) Use Frattini's argument: $N_G(P)\,\Phi(G)=G$, and deduce $N_G(P)=G$.]

5. Suppose G is a finite p-group; and $d(G)$ is the min. number of generators of G.

(a) Prove that $d(G) = \dim_{\mathbb{Z}/p\mathbb{Z}}\left(G/G^p[G,G]\right)$.

(b) Suppose S is a minimal generating set of G; that means $\langle S\rangle = G$ and $\langle S'\rangle \neq G$ if $S' \subsetneq S$.

Prove that $|S| = d(G)$.

(c) Does part (b) hold for finite groups that are not p-groups; that means for a finite group H do we have $|S_1| = |S_2|$ if S_1 and S_2 are two minimal generating sets?

[Hint (c) $\mathbb{Z}/6\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.]

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6. Prove that, if $G/_{Z(G)}$ is nilpotent, then G is nilpotent.

7. (a) Prove that $G/_{Z(G)}$ cannot be a non-trivial cyclic group.

(b) Prove that any group of order p^2 is abelian.

(c) Suppose G is a non-abelian group of order p^3. Prove that

(c1) $Z(G) \simeq \mathbb{Z}/_{p\mathbb{Z}}$.

(c2) $Z(G) = [G,G]$, and $G/_{Z(G)} \simeq \mathbb{Z}/_{p\mathbb{Z}} \times \mathbb{Z}/_{p\mathbb{Z}}$.

(c3) $d(G) = 2$; that means G can be gen. by 2 elements, but not by 1 element!