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It is not easy to understand if a given presentation gives us the trivial gp or not.

In fact one can ask the following question: given a presentation $G=\langle X \mid R\rangle$ and a word v. Can we give an algorithm to decide whether or not a given word w repre. the same element of G as v? This is called the word problem. Novikov proved that the answer is NO!

1. Prove that $\langle a,b \mid ab^2a^{-1}=b^3,\ ba^2b^{-1}=a^3\rangle$ is the trivial gp.

(Hint. Consider $a^2b^4a^{-2}$ and its conjugate by b. Deduce $a\in C_G(b^4)$.)

2. Prove that $\langle a,b \mid [a,b]\rangle \simeq \mathbb{Z}\oplus\mathbb{Z}$.

(Hint. As usual first find an onto gp hom. $\phi:\langle a,b\mid [a,b]\rangle \to \mathbb{Z}\oplus\mathbb{Z}$; Then consider $\theta:\mathbb{Z}\oplus\mathbb{Z}\to\langle a,b\mid[a,b]\rangle$, $\theta(m,n)=a^m b^n$.)

3. Suppose X_1 and X_2 are two disjoint sets of symbols. Prove that

$$\langle X_1 \mid R_1\rangle * \langle X_2 \mid R_2\rangle \simeq \langle X_1\sqcup X_2 \mid R_1\sqcup R_2\rangle.$$

(Hint. First define $\theta_i:\langle X_i\mid R_i\rangle \to \langle X_1\sqcup X_2 \mid R_1\sqcup R_2\rangle$;

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. Then define $\theta : \langle X_1 | R_1 \rangle * \langle X_2 | R_2 \rangle \longrightarrow \langle X_1 \sqcup X_2 | R_1 \sqcup R_2 \rangle$ using the universal property of the free product of two groups.

. Define $\phi : \langle X_1 \sqcup X_2 | R_1 \sqcup R_2 \rangle \longrightarrow \langle X_1 | R_1 \rangle * \langle X_2 | R_2 \rangle$.

. Check $\theta \circ \phi = id.$ and $\phi \circ \theta = id.$)

4. Prove that $G_n := \langle s_1, \ldots, s_{n-1} \mid s_i^2, (s_i s_j)^2 \text{ for } |i-j|>1, (s_i s_{i+1})^3 \rangle$ is isomorphic to S_n.

(Hint 1. Consider $\sigma_i := (i \;\; i+1)$ to get an onto group hom.
$\phi : G_n \rightarrow S_n$.

2. By induction on n, show $|G_n| \leq n!$; here is one way:

Let H_n be the subgp of G_n that is generated by $s_1, \ldots, s_{n-2}$.

$\boxed{2.a}$ Argue why there is an onto group hom. $G_{n-1} \rightarrow H_n$; and so by the induction hypothe. $|H_n| \leq (n-1)!$.

$\boxed{2.b}$ Show that

$$G_n / H_n = \{ H_n, s_{n-1} H_n, s_{n-2} s_{n-1} H_n, \ldots, s_1 s_2 \cdots s_{n-1} H_n \}. \quad ⊛$$

(And so $[G_n : H_n] \leq n$.) To show ⊛ check $s_i . RHS = RHS$.)

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5. ⓐ Prove that $\langle a,b \mid a^2, b^2 \rangle \simeq$ Group of (Euclidean) symmetries of $\mathbb{Z}$.

(Hint: • You can use without proof that the group of symmetries of $\mathbb{Z}$ is $G=\{f:\mathbb{Z}\to\mathbb{Z} \mid f(x)=cx+d,\ c\in\{\pm1\}, d\in\mathbb{Z}\}$; and it is generated by $f(x)=-x$ and $g(x)=x+1$.

• Consider f and $g\circ f$.

• Show $\langle a,b \mid a^2,b^2\rangle = \{(ab)^i \mid i\in\mathbb{Z}\} \cup \{a(ab)^i \mid i\in\mathbb{Z}\}$.)

ⓑ Prove that any group generated by <u>two</u> elements of order <u>2</u> is solvable.

6. Prove that $\left\langle \overline{\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}}, \overline{\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}} \right\rangle \simeq \mathbb{Z}/_{3\mathbb{Z}} * \mathbb{Z}/_{2\mathbb{Z}}$ where

$\overline{g} := g\{\pm I\} \in PSL(2,\mathbb{Z}) := SL_2(\mathbb{Z})/\{\pm I\}$.

(<u>Hint.</u> Consider the Möbius action of $PSL_2(\mathbb{R})$ on $\mathbb{R}\cup\{\infty\}$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot z = \frac{az+b}{cz+d}.$$

Let $\tau(z)=z+1$ and $\sigma(z)=\frac{-1}{z}$. Let $\omega := \tau\circ\sigma$. So $\omega(z) = -\frac{1}{z}+1$. Observe that $\omega^3(z)=z$. Consider $G_1=\langle\sigma\rangle$, $G_2=\langle\omega\rangle$, $X_1=(-\infty,0]$, $X_2=(0,\infty)\cup\{\infty\}$.)

(<u>Remark.</u> $PSL(2,\mathbb{Z}) = \left\langle \overline{\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}}, \overline{\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}} \right\rangle$.)

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7. The abelianization G^{ab} of the group G is $G^{ab} := G/_{[G,G]}$.

Notice that, if A is an abelian group and $\Phi : G \to A$ is a group homomorphism, then Φ factors through G^{ab}; that means

$\exists\, \overline{\Phi} : G^{ab} \to A$ s.t.
$$\begin{tikzcd} & G^{ab} \arrow[d, dashed, "\overline{\Phi}"] \\ G \arrow[ur, "\pi"] \arrow[r, "\Phi"'] & A \end{tikzcd}.$$

Prove that $(G * H)^{ab} \simeq G^{ab} \times H^{ab}$ for any two groups G & H.