Extra ring theory problems

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1. Let A be the subring of $\mathbb{Q}[x,y]$ which is generated by $x, xy, xy^2, \ldots$; that means $A = \mathbb{Q}[x, xy, xy^2, xy^3, \ldots]$.

Prove that A is NOT Noetherian.

2. A <u>Bezout</u> domain is an integral domain D in which

$$\forall a,b\in D,\ \exists c\in D \text{ s.t. } \langle a,b\rangle = \langle c\rangle.$$

(a) Prove that an integral domain D is a Bezout domain if and only if $\forall\, a,b\in D\setminus\{0\}$ $\exists d\in D$ s.t.

(i) d is a g.c.d. of a and b.

(ii) $d\in\langle a,b\rangle$.

(b) Prove that every finitely generated ideal of a Bezout domain is principal (In particular a Noetherian Bezout domain is a PID.)

(c) Prove that D is a PID if and only if it is both a UFD and a Bezout domain

(<u>Hint</u>. For $0\neq \mathfrak{a}\trianglelefteq D$, let $a\in\mathfrak{a}$ be an element with smallest number of irreducible factors.

$\forall b\in\mathfrak{a}$, show $\langle a,b\rangle = \langle a\rangle$.)

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3. Let D be a UFD and F be its field of fractions.

(a) Suppose $\frac{r}{s}$ is a zero of $a_n x^n + \cdots + a_1 x + a_0$ where $r, s \in D$ and $\gcd(r,s) = [1]$. Prove that $r \mid a_0$ and $s \mid a_n$.

In particular, if $a \in F$ is a zero of a monic poly. in $D[x]$, then $a \in D$. (We say a UFD is <u>integrally closed</u>.)

(b) Show that $\mathbb{Z}[2\sqrt{2}] = \{a + 2\sqrt{2}\, b \mid a, b \in \mathbb{Z}\}$ is not a UFD. (<u>Hint.</u> $(\sqrt{2})^2 - 2 = 0$.)

4. Let $A = \mathbb{Z} + x\mathbb{Q}[x] = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_0 \in \mathbb{Z}, a_1, \ldots, a_n \in \mathbb{Q}\}$.

(a) Show that $f(x) \in A$ is irreducible $\iff$

either $f(x) = \pm p$ where p is a prime number

or $f(x)$ is irred. in $\mathbb{Q}[x]$ and $f(0) = \pm 1$.

(b) Show that $x \in A$ cannot be written as a product of finitely many irreducibles in A. Thus A is not a UFD.

(c) We proved in class that, if an integral domain is Noetherian, then any non-zero element can be written as a product of

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irreducible. And A has an ideal that is not finitely generated. Find an explicit ideal $\mathfrak{a} \triangleleft A$ that is not finitely generated.

5. Suppose A is a unital commutative ring. For $I \triangleleft A$, let

$$\sqrt{I} := \{a \in A \mid \exists n \in \mathbb{Z}^+,\ a^n \in I\}.$$

(a) Prove that $\sqrt{I} \triangleleft A$ and $\mathrm{Nil}(A/I) = \sqrt{I}/I$.

(b) Prove that $\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \in \mathrm{Spec}(A) \\ I \subseteq \mathfrak{p}}} \mathfrak{p}$.

6. (Cohen) Suppose A is a unital commutative ring.

(a) Let $\Sigma := \{\mathfrak{a} \triangleleft A \mid \mathfrak{a} \text{ is not finitely generated}\}$. Suppose Σ is not empty. Prove that Σ has a maximal element.

(b) Suppose I is a maximal element of Σ. Prove that I is prime.

(Hint. Suppose to the contrary that I is not prime. So $\exists\, a, b$ s.t. $a, b \notin I$ and $ab \in I$. Deduce that $I + \langle a \rangle$ is finitely generated. Let $J := \{r \in A \mid r(I + \langle a \rangle) \subseteq I\}$; convience yourself that $J \triangleleft A$; and deduce that J is a finitely generated ideal. Find a generating set

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$\{f_1, \ldots, f_n, a\}$ for $I + \langle a \rangle$ s.t. $f_i \in I$. Show that $I = \langle f_1, \ldots, f_n \rangle + Ja$. Deduce that I is finitely generated, which is a contradiction.)

(c) Suppose any prime ideal of A is finitely generated. Prove that A is Noetherian.