Lecture 01: Symmetries of objects.

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Groups are symmetries of objects. What do we mean by a symmetry of an object X? Roughly it means a function $f: X \to X$ that preserves "properties" of X; and $f^{-1}: X \to X$ exists and preserves "properties" of X.

To understand this better, we look at a few examples:

At the level of set theory. When X is just a non-empty set, then any bijection $f: X \to X$ is a "symmetry" of X.

This group is denoted by S_X, and is called the symmetric group of X; $S_X := \{f: X \to X \mid f \text{ is a bijection}\}$.

For a positive integer n, we write S_n instead of $S_{\{1,\dots,n\}}$.

You have seen that $|S_n| = n!$.

Symmetries of a graph $\mathcal{G} = (V, E)$.

A symmetry of a graph $\mathcal{G}$ is a function $f: V \to V$ s.t. (1) f is a bijection (2) $\{v, w\} \in E \Leftrightarrow \{f(v), f(w)\} \in E$

(at the level of set theory.)

v is connected to w $\Leftrightarrow$ $f(v)$ is connected to $f(w)$.

Lecture 01: Dihedral group

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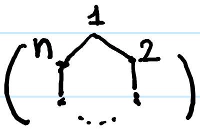
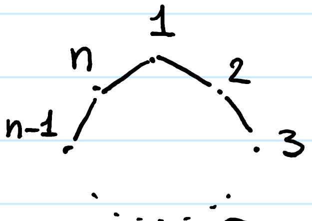
<u>The automorphism group of an n-cycle</u>. In this example, we would like to describe elements of Aut().

Typically in order to understand the group of symmetries of an object with "lots" of symmetries we use the following steps:

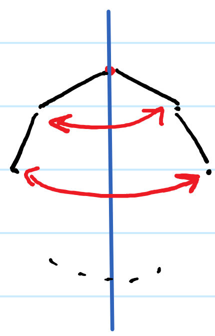
(1) Find a rich set of symmetries;

(2) Prove a type of "rigidity";

Can you give me a few symmetries of  ?

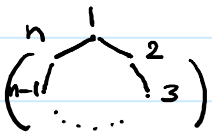
<u>"Rotations"</u>. Let $\tau \in S_n$, $1 \overset{\tau}{\mapsto} 2 \overset{\tau}{\mapsto} 3 \mapsto \cdots \overset{\tau}{\mapsto} n$

Then $\tau, \tau^2, \ldots, \tau^{n-1}$ are distinct.



<u>"Reflections"</u>. Let $\sigma \in S_n$,

$1 \mapsto 1$, $2 \mapsto n$, $3 \mapsto n-1$, $\ldots$.

So far we have found $\{ \mathrm{id.}, \tau, \ldots, \tau^{n-1}, \sigma, \tau\sigma, \ldots, \tau^{n-1}\sigma \} \subseteq$ Aut().

Lecture 01: Dihedral group

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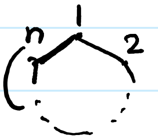
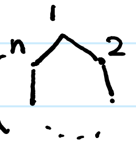
Next you can see the following "rigidity":

- An automorphism that fixes 1 and 2 is identity; prove by induction on i that $\gamma(i)=i$.

Now we show $\mathrm{Aut}($ 

$) = \{ id, \tau, \ldots, \tau^{n-1}, \sigma, \tau\circ\sigma, \ldots, \tau^{n-1}\circ\sigma \}$

Suppose $\gamma \in \mathrm{Aut}($ 

$)$. So $\exists\, i$ s.t. $\tau^{-i}\circ\gamma(1)=1$.

Since $\tau^{-i}\circ\gamma$ is an automorphism, $\tau^{-i}\circ\gamma(2)$ is connected to 1. Hence either $\tau^{-i}\circ\gamma(2)=2$ or $\tau^{-i}\circ\gamma(2)=n$.

Case 1. $\tau^{-i}\circ\gamma(2)=2$. Then, by rigidity, $\tau^{-i}\circ\gamma=id$.

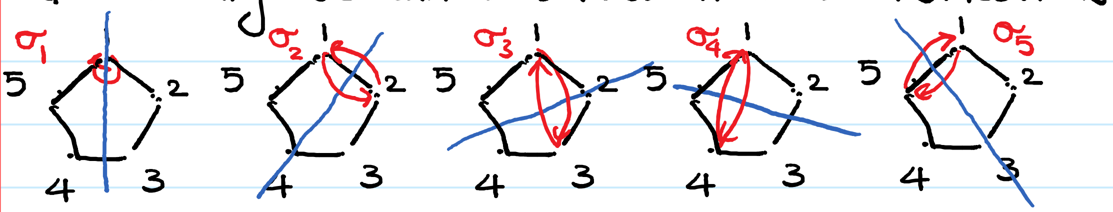
And so $\gamma=\tau^{i}$.

Case 2. $\tau^{-i}\circ\gamma(2)=n$. Then $\sigma\circ\tau^{-i}\circ\gamma$ fixes 1 and 2.

Hence, by rigidity, $\gamma=\tau^{i}\circ\sigma^{-1}=\tau^{i}\circ\sigma$.

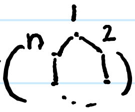
What happened to other reflections?

Geometrically we can construct n other reflections



So $\sigma_i(1)=\tau^i\circ\sigma(1)$ and $\sigma_i(2)=\tau^i\circ\sigma(2)$; hence by rigidity

$\sigma_i=\tau^i\circ\sigma$.

Aut() is called the <u>dihedral group D_{2n}</u>. So we just showed that D_{2n} has n rotations (including identity) and n reflections.

. What is the order of τ? $o(\tau)=n$.

. What is the order of σ (and $\tau^i\sigma$)? Since these are reflections, $o(\tau^i\sigma)=2$. In particular,

$\tau\sigma\tau\sigma=\mathrm{id}$. And so $\sigma\tau\sigma^{-1}=\tau^{-1}$.

. What is the order of τ^i? Recall that $o(\tau^i)=\dfrac{o(\tau)}{\gcd(i,o(\tau))}$,

and so $o(\tau^i)=\dfrac{n}{\gcd(i,n)}$.

<u>Symmetries of a metric space (X,d)</u>.

$f:X\to X$ is a symmetry if it is a bijection and

$d(x_1,x_2)=d(f(x_1),f(x_2))$ (it preserves distance)

Lecture 01: Group of isometries of the Euclidean plane

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Such a map is called an <u>isometry</u>.

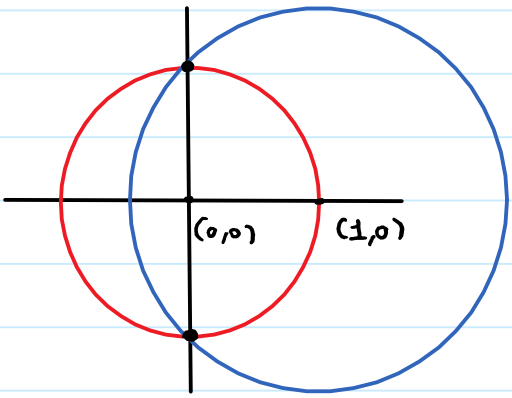
<u>Group of isometries of the Euclidean plane</u>. To understand this group we follow a method similar to the case of dihedral group.

<u>Lots of elements</u>. Rotations, reflections, and translations.

<u>Rigidity</u>. If an isometry fixes three points A, B, C that are not co-linear, then it is identity.

(A point D in a plane is uniquely determined by $|AD|$, $|BD|$, and $|CD|$.) [GPS works based on a similar observation.]

Suppose $\gamma \in \mathrm{Isom}(E)$. So $\exists$ a translation T s.t. $T^{-1} \circ \gamma(0,0) = (0,0)$. Then $\exists$ a rotation R centered at $(0,0)$ s.t. $R^{-1} \circ T^{-1} \circ \gamma(1,0) = (1,0)$.



Hence either $R^{-1} \circ T^{-1} \circ \gamma(0,1) = (0,1)$ or $R^{-1} \circ T^{-1} \circ (0,1) = (0,-1)$.

Therefore again by rigidity we deduce

Lecture 01: Group action

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$$\mathrm{Isom}(E) = \{ T\circ R \mid T \text{ is a translation},\ R \text{ is a rotation about } (0,0) \}$$
$$\cup \{ T\circ R\circ L \mid T \text{ is a translation},\ R \text{ is a rotation about } (0,0),\ L(x,y)=(x,-y) \text{ the reflection about the } x\text{-axis} \}$$

Identifying E with $\mathbb{R}^2$ and using linear algebra, we get that
$$\mathrm{Isom}(E) = \{ f:\mathbb{R}^2\to\mathbb{R}^2 \mid f(v)=Kv+b,\ K \text{ orthogonal } 2\times 2 \text{ matrix},\ b\in\mathbb{R}^2 \}.$$

- Next we start with an abstract group G and an object X and try to view G as possible symmetries of X.

Def. Let G be a group, and X be a non-empty set. A (left) action of G on X is $m: G\times X\to X$, $m(g,x):=g\cdot x$ which has the following properties:

(1) $\forall x\in X$, $e\cdot x = x$ where e is the neutral element of G.

(2) $\forall x\in X$, $\forall g_1,g_2\in G$, $g_1\cdot(g_2\cdot x) = (g_1g_2)\cdot x$.

We say G acts on X, and write $G\curvearrowright X$.

Lecture 01: Group actions and group of symmetries

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Suppose X is an object; think about a set, a graph, Euclidean plane, a vector space, a group, a ring, etc. Then

$\mathrm{Symm}(X) \curvearrowright X$.

Let's try to guess what the action is. Recall that

$$\mathrm{Symm}(X) := \left\{ f: X \to X \;\middle|\; \begin{array}{l} (1)\ f \text{ is a bijection} \\ (2)\ f \text{ and } f^{-1} \text{ preserve} \\ \quad\ \text{structure of } X \end{array} \right\}.$$

We need to define $m: \mathrm{Symm}(X) \times X \to X$.

$$(f, x) \mapsto ?$$

The group action should tell us what the group element f does to the point x. As soon as we phrase the question in this way, we automatically answer $f(x)$. So let

$$m: \mathrm{Symm}(X) \times X \to X, \quad m(f,x) := f(x).$$

Why is it a group action?

$m(\mathrm{id.}, x) = \mathrm{id.}(x) = x$ ✓

$f_1 \cdot (f_2 \cdot x) = f_1(f_2(x)) = (f_1 \circ f_2)(x) = (f_1 \circ f_2) \cdot x$ ✓.

Lecture 01: Some examples of group actions

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The above example immediately gives us a lot of interesting examples:

- $S_n \curvearrowright \{1,2,\ldots,n\}$ $\quad (\sigma, i) \mapsto \sigma(i)$
- (Vector space $\mathbb{R}^n$) $\underbrace{GL_n(\mathbb{R})}_{\substack{n\times n \text{ invertible} \\ \text{real matrices}}} \curvearrowright \mathbb{R}^n$, $(A, v) \mapsto Av$
- (A group G) A "symmetry" of a group is called an automorphism.

$$\mathrm{Aut}(G) = \left\{ \phi: G \to G \;\middle|\; \begin{array}{l} (1)\ \phi \text{ is a bijection} \\ (2)\ \phi, \phi^{-1} \text{ preserve} \\ \quad\ \text{structure of } G \end{array} \right\}.$$

To understand (2), let's recall what a group homomorphism is.

<u>Def.</u> Suppose H_1 and H_2 are two groups; then $f: H_1 \to H_2$ is called a group homomorphism if

(a) $f(e_{H_1}) = e_{H_2}$

(b) $f(h\,h') = f(h)\, f(h')$ $\quad \forall h, h' \in H$

(c) $f(h^{-1}) = f(h)^{-1}$ $\quad \forall h \in H$

Let $\mathrm{Hom}(H_1, H_2) := \{ f: H_1 \to H_2 \mid f \text{ is a group hom.} \}$

Lecture 01: Parametrizing group actions

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<u>Exercise</u> (b) implies (a) and (c).

<u>Exercise</u> If $\phi \in \mathrm{Hom}(H_1, H_2)$ is bijective, then $\phi^{-1} \in \mathrm{Hom}(H_2, H)$.

So $\mathrm{Aut}(G) = \{\phi : G \to G \mid \phi$ is a bijection and $\forall g, g' \in G,\ \phi(gg') = \phi(g)\phi(g')\}$;

and $\mathrm{Aut}(G) \curvearrowright G$, $(\phi, g) \mapsto \phi(g)$.

Next we would like to parametrize all the possible group actions of a group G on a set X (the same can be done for any object). This means to find out what functions $m : G \times X \to X$ give us a group action.

One can think of such a function as <u>a family of functions from X to X that is indexed over G</u>:

$$m : G \times X \to X \quad \rightsquigarrow \quad \{m_g\}_{g \in G}, \quad m_g : X \to X, \quad m_g(x) := m(g, x).$$

$$\mathrm{Functions}\,(G \times X, X) \rightsquigarrow \mathrm{Functions}\,(G, \mathrm{Functions}(X, X)).$$

is a bijection.

Lecture 01: Parametrizing group actions

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Next we show

Theorem. Let G be a group and X be a non-empty set.

Let $\mathrm{act}(G,X) := \{ m : G\times X \to X \mid m : \text{a left group action}\}$.

Then $\Psi : \mathrm{act}(G,X) \to \mathrm{Hom}(G, S_X)$,

$$((\Psi(m))(g))(x) := m(g,x)$$

and $\Phi : \mathrm{Hom}(G, S_X) \to \mathrm{act}(G,X)$,

$$\Phi(f)(g,x) := (f(g))(x)$$

are well-defined and inverse of each other.

We will go over this proof next time

Pf. • Step 1. $\left((\Psi(m))(g_1g_2)\right)(x) = (g_1g_2)\cdot x$

$$= g_1\cdot(g_2\cdot x)$$

$$= (\Psi(m))(g_1)(g_2\cdot x)$$

$$= \left((\Psi(m))(g_1)\circ(\Psi(m))(g_2)\right)(x)$$

And so $(\Psi(m))(g_1g_2) = (\Psi(m))(g_1)\circ(\Psi(m))(g_2)$.

• Step 2. $\left((\Psi(m))(e)\right)(x) = e\cdot x = x$

And so $(\Psi(m))(e) = \mathrm{id}_X$.

• Step 3. Step 1 and 2 imply $\Psi(m)(g^{-1})\circ\Psi(m)(g) = \mathrm{id}_X$;

Lecture 01: Parametrizing group actions

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and so $(\Psi(m))(g) \in S_X$ for any $g \in G$. Therefore,

by Step 1, $\Psi(m) \in \mathrm{Hom}(G, S_X)$.

- Step 4. $g \cdot x := (f(g))(x)$ for $f \in \mathrm{Hom}(G, S_X)$.

  Then $f(e) = \mathrm{id}_X$; and so $e \cdot x = (f(e))(x) = x$.

  $$(g_1 g_2) \cdot x = f(g_1 g_2)(x) = (f(g_1) \circ f(g_2))(x)$$
  $$= f(g_1)(f(g_2)(x)) = g_1 \cdot (g_2 \cdot x).$$

  Hence Φ is well-defined.

- Step 5. $[(\Phi \circ \Psi)(m)](g, x)$

  $$= \Phi(\Psi(m))(g, x) = ((\Psi(m))(g))(x) = m(g, x)$$

  Thus $\Phi \circ \Psi = \mathrm{id}$.

- Step 6. $(((\Psi \circ \Phi)(f))(g))(x) =$

  $$(\Psi(\Phi(f))(g))(x) =$$
  $$\Phi(f)(g, x) =$$
  $$(f(g))(x);$$ and so $\Psi \circ \Phi = \mathrm{id}$. ∎

We will go over this proof next time.

Lecture 01: Left translation and Cayley's theorem

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Ex. (The left translation action) $G \curvearrowright G$ by the left translation; that means $g \cdot x := gx$.

By the previous theorem this action corresponds to a group homomorphism $f: G \to S_G$, $(f(g))(x) := gx$. Cayley's theorem that f is injective and so any group can be embedded into a symmetric group.

Theorem (Cayley) A group G can be embedded into the symmetric group S_G.

Pf 1. By the above argument $f: G \to S_G$, $(f(g))(x) := gx$ is a group homomorphism. So it is enough to show f is injective; that means $\ker f = \{e\}$.

Suppose $f(g) = \mathrm{id}$. Then $(f(g))(e) = e$; and so $g = ge = e$.

Pf 2. (Since this is an important result, we give a self-contained argument which is the same as above.)

Lecture 01: Cayley and left translation

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For $\forall g \in G$, let $l_g : G \to G$, $l_g(g') := gg'$;

Step 1. $(l_{g_1} \circ l_{g_2})(g) = l_{g_1}(l_{g_2}(g))$

$$= g_1(g_2 g) = (g_1 g_2) g$$

$$= l_{g_1 g_2}(g)$$

And so $l_{g_1} \circ l_{g_2} = l_{g_1 g_2}$.

Step 2. $l_e(g) = eg = g$; and so $l_e = \mathrm{id}$.

Step 3. By Steps 1 and 2, $l_{g^{-1}}$ is the inverse of l_g; and so $l_g \in S_G$.

Step 4. By Steps 1 and 3, $g \mapsto l_g$ is a group hom $f : G \to S_G$.

Step 5. $g \in \ker f \Rightarrow f(g) = \mathrm{id}_G \Rightarrow (f(g))(e) = e$

$$\Rightarrow g = ge = e. \quad \blacksquare$$

Ex. (The left translation on cosets) Suppose G is a group and H is a subgroup. Then $G \curvearrowright G/H$ by left translations; that means $g \cdot (g'H) := gg'H$.

We will discuss the 2nd pf in the next lecture.