Lecture 02: Cayley's theorem

Tuesday, October 2, 2018 8:45 AM

At the end of the previous lecture we proved Cayley's theorem. Since this is an important result, we give a self contained proof.

<u>Theorem</u> (Cayley) Let G be a group. Then $G \hookrightarrow S_G$.

<u>Pf</u>. Let $f: G \to S_G$, $f(g) := l_g$ where

$$l_g : G \to G,\ l_g(g') := gg'.$$

<u>Step 1</u>. $f(g_1 g_2) = f(g_1) \circ f(g_2)$.

$$(l_{g_1} \circ l_{g_2})(g') = l_{g_1}(l_{g_2}(g')) = g_1 (g_2 g')$$
$$= (g_1 g_2) g' = l_{g_1 g_2}(g').$$

<u>Step 2</u>. $f(e) = \mathrm{id}_G$.

$$l_e(g') = eg' = g'.$$

<u>Step 3</u>. $f(g) \in S_G$.

$$l_g \circ l_{g^{-1}} = l_{gg'} = l_e = \mathrm{id}_G; \text{ similarly } l_{g^{-1}} \circ l_g = \mathrm{id}_G$$

Hence $f(g) = l_g \in S_G$.

<u>Step 4</u>. $f: G \to S_G$ is an injective group hom.

Step 1 and Step3 imply f is a group hom.

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To show f is injective, it is enough to show its kernel

$$\ker f := \{g \in G \mid f(g) = \mathrm{id}_G\}$$

is trivial:

$$f(g) = \mathrm{id} \Rightarrow f(g)(e) = e \Rightarrow ge = e \Rightarrow g = e.$$

■

In the previous lecture we mentioned also the following theorem.

<u>Theorem</u> (Parametrizing group actions)

Let G be a group and X be a non-empty set. Let

$\mathrm{Act}(G,X) := \{m : G \times X \to X \mid m \text{ gives us a group action}\}$.

Let $\Psi : \mathrm{Act}(G,X) \to \mathrm{Hom}(G, S_X)$

$$((\Psi(m))(g))(x) := m(g,x), \quad \text{and}$$

$$\Phi : \mathrm{Hom}(G,X) \to \mathrm{Act}(G,X)$$

$$(\Phi(f))(g,x) := (f(g))(x).$$

Then Ψ and Φ are inverse of each other.

<u>Outline of proof.</u> We fix $m \in \mathrm{Act}(G,X)$ and would like to show

$\Psi(m) \in \mathrm{Hom}(G, S_X)$. Let $\xi(g) := (\Psi(m))(g)$.

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First we think about $\xi(g)$ just as a function from X to X.

Step 1. $\forall g_1, g_2 \in G$, $\xi(g_1 g_2) = \xi(g_1) \circ \xi(g_2)$.

Pf. $\xi(g_1 g_2)(x) = (g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) = \xi(g_1)\big(\xi(g_2)(x)\big)$

$$= \big(\xi(g_1) \circ \xi(g_2)\big)(x).$$

Step 2. $\xi(e) = \mathrm{id}_X$.

Step 3. $\xi(g) \in S_X$ (Show $\xi(g) \circ \xi(g^{-1}) = \xi(g^{-1}) \circ \xi(g) = \mathrm{id}_X$.)

Step 4. $\xi : G \to S_X$ is a well-defined gp hom.

Next we fix $f \in \mathrm{Hom}(G, S_X)$, let $g \cdot x := f(g)(x)$. We will show this is a group action.

$g_1 \cdot (g_2 \cdot x) = f(g_1)\big(f(g_2)(x)\big) = \big(f(g_1) \circ f(g_2)\big)(x) = f(g_1 g_2)(x)$

$$= (g_1 g_2) \cdot x$$

$e \cdot x = f(e)(x) = \mathrm{id}_X(x) = x$.

Ex. Check Φ and Ψ are inverse of each other. $\square$

(You can look at lecture 1 notes).

Lecture 02: An important trick

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The following is a nice trick based on the previous theorem.

- Suppose G is a "very large" group, X is a "small" set, and $G \curvearrowright X$ non-trivially. Then G is not simple.

<u>Pf.</u> $G \curvearrowright X$ has a (non-trivial) associate group hom. $f: G \to S_X$. Since $|G|$ is "very large" and X is "small" (we need $|G| > |X|!$), $\ker f \neq \{e\}$; and so G has a non-trivial normal subgp. ■

The following special case is interesting:

- Suppose G is a "large group" and it has a subgp H with small index; to be precise suppose $|G| > [G:H]!$. Then G is not simple.

<u>Pf.</u> Just use the above trick for the left translation action $G \curvearrowright G/H$. ■

Lecture 02: Induced group action

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<u>Lemma</u> (Induced group action) Suppose $G \curvearrowright X$ and $\theta \in \mathrm{Hom}(H,G)$. Then the following defines a left group action of H on X: $h * x := \theta(h)\cdot x$.

<u>Pf 1</u> By the previous theorem, $\Psi(m) \in \mathrm{Hom}(G, S_X)$ where $\big((\Psi(m))(g)\big)(x) = g\cdot x$. Hence $\Psi(m)\circ\theta \in \mathrm{Hom}(H, S_X)$. And so $\Phi(\Psi(m)\circ\theta)$ is in $\mathrm{act}(H,X)$; and

$$\begin{aligned}\big(\Phi(\Psi(m)\circ\theta)\big)(h,x) &= \big((\Psi(m)\circ\theta)(h)\big)(x)\\ &= \big(\Psi(m)(\theta(h))\big)(x)\\ &= \theta(h)\cdot x.\end{aligned}$$

<u>Pf 2</u>. $\cdot$ $e_H * x = \theta(e_H)\cdot x = e_G\cdot x = x$

$\cdot$ $h_1 * (h_2 * x) = \theta(h_1)\cdot(\theta(h_2)\cdot x) = (\theta(h_1)\,\theta(h_2))\cdot x$

$$= \theta(h_1h_2)\cdot x = (h_1h_2) * x. \qquad \blacksquare$$

<u>Ex.</u> $G \curvearrowright G$ by conjugation; that means $g * g' := g g' g^{-1}$.

<u>Pf.</u> $g_1 * (g_2 * g') = g_1 (g_2 g' g_2^{-1}) g_1^{-1} = (g_1 g_2) g' (g_1 g_2)^{-1} = (g_1 g_2) * g'$.

Lecture 02: Conjugation

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$e * g' = e g' e^{-1} = g'$.

<u>Outline of 2^{nd} approach</u>. By the meta-example proved in the 1^{st} lecture, $\mathrm{Aut}(G) \curvearrowright G$; on the other hand, as you will prove in your HW, $c: G \to \mathrm{Aut}(G)$, $(c(g))(g') := g g' g^{-1}$ is a group homomorphism. And so we get an induced group action $G \curvearrowright G$. Going through the bijections, one can see that this is the conjugation action. ∎

<u>Ex</u>. $G \curvearrowright \{H \mid H \leq G\}$; $G \curvearrowright \{H \mid H \leq G, [G:H] = n\}$.

<u>Pf</u>. $c(g) \in \mathrm{Aut}(G)$; and so $c(g)(H) \leq G$; hence $gHg^{-1} \leq G$. By a similar argument as in the previous example, we get the 1^{st} claim. To see the 2^{nd} claim it is enough to show $[G : gHg^{-1}] = [G:H]$.

(In general, if $G \curvearrowright X$, $\emptyset \neq Y \subseteq X$, $\forall g \in G, y \in Y, g \cdot y \in Y$, then $G \curvearrowright Y$.)

$[G:H] = n$ implies $\exists g_i$ s.t. $G = \coprod_{i=1}^{n} H g_i$; and so

Lecture 02: Orbits

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$$G = c(g)(G) = \bigsqcup_{i=1}^{n} c(g)(H g_i) = \bigsqcup_{i=1}^{n} c(g)(H)\, c(g)(g_i)$$

$$= \bigsqcup_{i=1}^{n} (gHg^{-1})(gg_ig^{-1}) \text{; thus } [G : gHg^{-1}] = n. \quad \blacksquare$$

<u>Def.</u> • Suppose $G \curvearrowright X$, $x \in X$. The the G-orbit of x is

$$G\cdot x := \{g\cdot x \mid g \in G\}.$$

• The stabilizer of x is $G_x := \{g \in G \mid g\cdot x = x\}$.

<u>Lemma</u>. G_x is a subgroup of G.

<u>Pf.</u> • $e\cdot x = x \Rightarrow e \in G_x$

• $g_1, g_2 \in G_x \Rightarrow (g_1g_2)\cdot x = g_1\cdot(g_2\cdot x) = g_1\cdot x = x$
$\Rightarrow g_1g_2 \in G_x$

• $g \in G_x \Rightarrow g\cdot x = x \Rightarrow g^{-1}\cdot(g\cdot x) = g^{-1}\cdot x$
$\Rightarrow (g^{-1}g)\cdot x = g^{-1}\cdot x \Rightarrow e\cdot x = g^{-1}\cdot x$
$\Rightarrow x = g^{-1}\cdot x \Rightarrow g^{-1} \in G_x.$ $\blacksquare$

<u>Lemma</u>. Suppose $G \curvearrowright X$. Then the following are equivalent.

(1) $G\cdot x_1 = G\cdot x_2$, (2) $x_2 \in G\cdot x_1$, (3) $G\cdot x_1 \cap G\cdot x_2 \neq \emptyset$.

<u>Pf.</u> (1) $\Rightarrow$ (2) $x_2 = e\cdot x_2 \in G\cdot x_2 = G\cdot x_1$

(2) $\Rightarrow$ (3) $(x_2 \in G\cdot x_1,\ x_2 = e\cdot x_2 \in G\cdot x_2) \Rightarrow x_2 \in G\cdot x_1 \cap G\cdot x_2$.

Lecture 02: Quotient space

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(3) $\Rightarrow$ (1) $y \in G\cdot x_1 \cap G\cdot x_2 \Rightarrow \exists g_1, g_2 \in G,\ g_1\cdot x_1 = g_2\cdot x_2$

$$\Rightarrow x_1 = (g_1^{-1} g_2)\cdot x_2$$

$$\Rightarrow \forall g \in G,\ g\cdot x_1 = (g g_1^{-1} g_2)\cdot x_2 \in G\cdot x_2$$

$$\Rightarrow G\cdot x_1 \subseteq G\cdot x_2 \text{ ; and by symmetry } G\cdot x_2 \subseteq G\cdot x_1 .$$

■

Theorem. $G\backslash X := \{ G\cdot x \mid x \in X \}$ is a partition of X.

Pf. $\forall x \in X$, $x = e\cdot x \in G\cdot x$; and so $\bigcup_{x\in X} G\cdot x = X$.

And by the previous lemma, $G\cdot x$'s are disjoint; and claim follows. ■

Theorem (Orbit-Stabilizer) The following is a bijection:

$$G/G_x \xrightarrow{f} G\cdot x, \quad gG_x \longmapsto g\cdot x .$$

Pf. Well-defined. $g_1 G_x = g_2 G_x \Rightarrow g_2 = g_1 h$ for some $h \in G_x$

$$\Rightarrow g_2\cdot x = (g_1 h)\cdot x = g_1\cdot(h\cdot x) = g_1\cdot x$$

Injective. $g_1\cdot x = g_2\cdot x \Rightarrow (g_2^{-1} g_1)\cdot x = x$

$$\Rightarrow g_2^{-1} g_1 \in G_x \Rightarrow g_1 G_x = g_2 G_x .$$

Surjective. $y \in G\cdot x \Rightarrow \exists g \in G,\ y = g\cdot x = f(gG_x)$. ■

Lecture 02: Consequence of orbit-stabilizer theorem

Wednesday, October 3, 2018 10:38 PM

<u>Theorem</u>. Suppose G is a finite group, X is a non-empty finite set, and $G \curvearrowright X$. Then

$$\frac{|X|}{|G|} = \sum_{G\cdot x \in {}_G\backslash X} \frac{1}{|G_x|}.$$

<u>Pf.</u> Since ${}_G\backslash X$ is a partition of X,

$$|X| = \sum_{G\cdot x \in {}_G\backslash X} |G\cdot x|.$$

By the orbit-stabilizer theorem, $|G\cdot x| = [G:G_x]$.

Hence $|X| = \sum_{G\cdot x \in {}_G\backslash X} [G:G_x] = \sum_{G\cdot x \in {}_G\backslash X} \frac{|G|}{|G_x|}$. ∎