Lecture 03: Fixed points of an element

Wednesday, October 3, 2018 10:47 PM

In the previous lecture we showed $G/G_x \to G\cdot x$, $gG_x \mapsto g\cdot x$ is a bijection, and ${}_G\backslash X := \{G\cdot x \mid x\in X\}$ is a partition of X.

<u>Lemma</u>. $\forall g\in G$, $x\in X$, $G_{g\cdot x} = g G_x g^{-1}$; in particular if $|G_x|<\infty$, then $|G_{g\cdot x}| = |G_x|$.

<u>Pf</u>. $h\in G_{g\cdot x} \Leftrightarrow h\cdot(g\cdot x) = g\cdot x$

$\Leftrightarrow (g^{-1}hg)\cdot x = x$

$\Leftrightarrow g^{-1}hg\in G_x \Leftrightarrow h\in gG_xg^{-1}$. ∎

<u>Def</u>. Suppose $G \curvearrowright X$; $\forall g\in G$, $X^g := \{x\in X \mid g\cdot x = x\}$

(the set of fixed points of g.)

<u>Lemma</u>. $\forall g,h\in G$, $g\cdot X^h = X^{ghg^{-1}}$.

<u>Pf</u>. $x\in X^{ghg^{-1}} \Leftrightarrow (ghg^{-1})\cdot x = x$

$\Leftrightarrow h\cdot(g^{-1}\cdot x) = g^{-1}\cdot x$

$\Leftrightarrow g^{-1}\cdot x\in X^h \Leftrightarrow x\in g\cdot X^h$. ∎

<u>Cor</u>. $|X^h| = |X^{ghg^{-1}}|$ $\forall g,h\in G$.

Lecture 03: Lemma that is not Burnside's

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Theorem. $G \curvearrowright X$, $|G|, |X| < \infty$. Then

$$|{}_G\backslash^X| = \frac{1}{|G|} \sum_{g\in G} |X^g|$$

(The number of elements of the quotient space is the average of the number of fixed points of elements of G.)

Pf. Let $A := \{(g,x) \in G\times X \mid g\cdot x = x\}$. Then

$$|A| = \sum_{g\in G} |X^g| = \sum_{x\in X} |G_x|.$$

$$= \sum_{G\cdot x \in {}_G\backslash^X} \sum_{y\in G\cdot x} |G_y|$$

Since $y \in G.x$, G_y is a conjugate of G_x

$$= \sum_{G\cdot x \in {}_G\backslash^X} \sum_{y\in G\cdot x} |G_x|$$

$$= \sum_{G\cdot x \in {}_G\backslash^X} |G_x||G\cdot x|$$

Orbit-Stabilizer theorem

$$= \sum_{G\cdot x \in {}_G\backslash^X} |G|$$

$$= |G|\,|{}_G\backslash^X|$$; and claim follows. ∎

Def. We say $G \curvearrowright X$ is <u>transitive</u> if $X = G\cdot x$.

Lecture 03: Transitive actions

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<u>Proposition</u>. Suppose $G \curvearrowright X$ is transitive and $|G|<\infty$, $|X|>1$.

Then $\exists g\in G$ s.t. $X^g=\emptyset$.

<u>Pf.</u> Suppose to the contrary that $X^g \neq \emptyset$ for any $g\in G$. Hence $|X^g|\geq 1$ $\forall g\in G$. So by Lemma that is not Burnside's we have

$$|{}_G\backslash X| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$

$$=\frac{1}{|G|}\Big(|X^e|+\sum_{g\in G\setminus\{e\}}|X^g|\Big)$$

$$\geq \frac{1}{|G|}\big(|X|+|G|-1\big).$$

Since $G\curvearrowright X$ is transitive, $|{}_G\backslash X|=1$; and so

$1\geq \frac{1}{|G|}(|X|+|G|-1)$ which implies $|X|\leq 1$; and that is a contradiction. ∎

<u>Problem</u>. Suppose G is a finite group, $H \lneq G$. Prove that

$$G\neq \bigcup_{g\in G} gHg^{-1}.$$

<u>Solution</u>. $G\curvearrowright G/H$ by left translations transitively $\Rightarrow$

$\exists g_0\in G$, $(G/H)^{g_0}=\emptyset$.

Lecture 03: Quotient space as a partition

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$$\underbrace{(G/H)}_{X}^{g_0} = \emptyset \Rightarrow g_0 \notin \bigcup_{x \in X} G_x \,. \quad (*)$$

$g' \in$ The stabilizer of $gH \iff g'gH = gH$

$$\iff g^{-1} g' g \in H$$

$$\iff g' \in g H g^{-1}$$

So stabilizer of $gH = gHg^{-1}$. Hence $(*)$ implies

$$g_0 \notin \bigcup_{g \in G} g H g^{-1} . \qquad \blacksquare$$

<u>Def.</u> $X^G := \{x \in X \mid \forall g \in G,\ g \cdot x = x\}$. (The set of fixed points of G).

Notice that $x \in X^G \iff |G \cdot x| = 1$. So

$$|X| = \sum_{G \cdot x \in X/G} |G \cdot x| = |X^G| + \sum_{\substack{G \cdot x \in X/G \\ |G \cdot x| > 1}} |G \cdot x|$$

$$\Rightarrow \boxed{|X| = |X^G| + \sum_{\substack{G \cdot x \in X/G \\ x \notin X^G}} [G : G_x] \quad \text{by Orbit-Stabilizer.}}$$

Lecture 03: Conjugacy classes and class equation

Thursday, October 4, 2018 12:19 AM

Let's study $G \curvearrowright G$ by conjugation.

G-orbit of g is $\{g' g g'^{-1} \mid g' \in G\}$; this is called the <u>conjugacy class of g</u>, and we denote it by $Cl(g)$.

<u>The stabilizer of</u> $g = \{g' \in G \mid g' g g'^{-1} = g\}$

$= \{g' \in G \mid g' g = g g'\}$

$= C_G(g)$ is called the centralizer of g.

So $|Cl(g)| = [G : C_G(g)]$ by orbit-stabilizer theorem.

The set of fixed points of $G = \{g \in G \mid \forall g' \in G, g' g g'^{-1} = g\}$

$= \{g \in G \mid \forall g' \in G, g' g = g g'\}$

$= Z(G)$ is called the center of G.

By the previous equation, we have

$$\boxed{|G| = |Z(G)| + \sum_{\substack{\text{representative} \\ \text{of conjugacy} \\ \text{classes;} \\ \text{not in } Z(G)}} [G : C_G(g)]}$$

This is called the class equation.

Lecture 03: Kernel of a group action and normal core

Thursday, October 4, 2018 12:35 AM

Def. Suppose $G \curvearrowright X$. Kernel of this action is

$$\{g \in G \mid \forall x \in X,\ g \cdot x = x\}.$$

$\boxed{\text{Important}}$ If $f: G \to S_X$ is the group homomorphism associated with $G \curvearrowright X$, then the kernel of the group action is $\ker(f)$; and so it is a normal subgroup of G and by the 1st isomorphism theorem $G/\ker f \hookrightarrow S_X$.

Let's study $G \curvearrowright G/H$ by the left translations.

- Action is transitive.
- Stabilizer of gH is gHg^{-1}.
- Kernel of this action is $\bigcap_{g \in G} gHg^{-1}$. This is called the normal core of H and we denote it by $\mathrm{cor}(H)$.

Hence $G/\mathrm{cor}(H) \hookrightarrow S_{(G/H)}$; in particular,

$$[G : \mathrm{cor}(H)] \mid [G:H]!\ .$$

Lemma. Suppose $H \leq G$, $N \trianglelefteq G$, and $N \subseteq H$. Then

$$N \subseteq \mathrm{cor}(H).$$

Lecture 03: Normal core of a subgroup

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Pf. $N \subseteq H \Rightarrow gNg^{-1} \subseteq gHg^{-1}$

$$\overset{\parallel}{N} \subseteq gHg^{-1} \Rightarrow N \subseteq \bigcap_{g\in G} gHg^{-1} = \mathrm{cor}(H).$$

■

So cor(H) is the largest normal subgroup of G that is contained in H.

Problem. Suppose G is a finite group, $H \leq G$, $[G:H]=p$, where p is the smallest prime factor of $|G|$. Prove that $H \triangleleft G$.

Solution. By the previous discussion,

$$[G:\mathrm{cor}(H)] \mid [G:H]! = p!.$$

And $[G:\mathrm{cor}(H)] \mid |G|$. Hence $[G:\mathrm{cor}(H)] \mid \gcd(|G|, p!)$

Since p is the smallest prime factor of $|G|$, $\gcd(|G|, p!) = p$.

Therefore $[G:\mathrm{cor}(H)] \mid p$, which implies $[G:\mathrm{cor}(H)] = p$.

As $\mathrm{cor}(H) \subseteq H$ and $[G:H] = p = [G:\mathrm{cor}(H)]$, we deduce

$H = \mathrm{cor}(H) \triangleleft G$. ■

Lecture 03: Normalizer

Thursday, October 4, 2018 1:11 AM

$G \curvearrowright \{H \mid H \leq G\}$ by conjugation.

Stab. of $H = \{g \in G \mid gHg^{-1} = H\}$

is called <u>the normalizer of H in G</u>

and it is denoted by $N_G(H)$.

Orbit of $H = \{gHg^{-1} \mid g \in G\}$.

So # of conjugates of $H = [G : N_G(H)]$.

. Notice that $N_G(H)$ is the largest subgroup of G which has H as a normal subgroup.

Next we prove an extremely useful result about actions of p-groups.

<u>Theorem</u>. Suppose $|G| = p^n$ where p is prime. And $G \curvearrowright X$, $|X| < \infty$. Then $|X| \equiv |X^G| \pmod p$.

We will prove this in the next lecture, and then use it to prove Sylow theorems.