Lecture 09: Sign function

Thursday, October 25, 2018 8:38 AM

Recall. $\Delta(x_1,\dots,x_n) := \prod_{i<j}(x_i - x_j)$ and

$$\Delta_\sigma(x_1,\dots,x_n) := \Delta(x_{\sigma(1)},\dots,x_{\sigma(n)}) = \prod_{i<j}(x_{\sigma(i)} - x_{\sigma(j)}).$$

We showed $\prod_{i\neq j}(x_i - x_j) = (-1)^{\frac{n(n-1)}{2}}\Delta^2$; and so

$$(-1)^{\frac{n(n-1)}{2}}\Delta_\sigma^2 = \prod_{i\neq j}(x_{\sigma(i)} - x_{\sigma(j)}) = \prod_{i\neq j}(x_i - x_j)$$

$$= (-1)^{\frac{n(n-1)}{2}}\Delta^2.$$

Therefore $\exists\, \epsilon : S_n \to \{\pm 1\}$, $\Delta_\sigma = \epsilon(\sigma)\,\Delta$.

Lemma. ϵ is a group homomorphism.

Pf. $\Delta_{\sigma\tau}(x_1,\dots,x_n) = \Delta(x_{\sigma(\tau(1))},\dots,x_{\sigma(\tau(n))})$

$$= \Delta_\tau(x_{\sigma(1)},\dots,x_{\sigma(n)})$$

$$\boxed{\begin{array}{c} y_i = x_{\sigma(i)} \Rightarrow \Delta_\tau(y_1,\dots,y_n) = (y_{\tau(1)},\dots,y_{\tau(n)}) \\ \text{and } y_{\tau(j)} = x_{\sigma(\tau(j))}. \end{array}}$$

$$= \epsilon(\tau)\,\Delta(x_{\sigma(1)},\dots,x_{\sigma(n)})$$

$$= \epsilon(\tau)\,\epsilon(\sigma)\,\Delta.$$

$\Rightarrow \epsilon(\sigma\tau)\Delta = \epsilon(\tau)\epsilon(\sigma)\,\Delta \Rightarrow \epsilon(\sigma\tau) = \epsilon(\sigma)\epsilon(\tau)$. ∎

Def. ϵ is called the sign function;

Lecture 09: Number of crossings

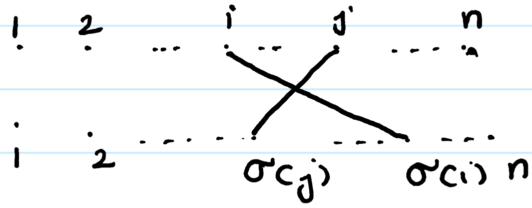
Thursday, October 25, 2018 8:56 AM

- Sometimes ϵ is denoted by sgn.

Notice that $\Delta_\sigma = \prod_{i<j} (x_{\sigma(i)} - x_{\sigma(j)})$

$= (-1)^{n_\sigma} \Delta$ where

$n_\sigma := \#\{(i,j) \mid i<j \text{ and } \sigma(i) > \sigma(j)\}$; so n_σ is the # of crossings in



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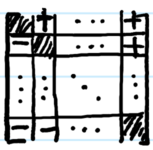
In the theory of root systems it is proved that $n_\sigma = \ell(\sigma)$ where $\ell(\sigma)$ is the word metric of σ w.r.t. the generating set $\{(1\ 2), (2\ 3), \dots, (n-1\ n)\}$.

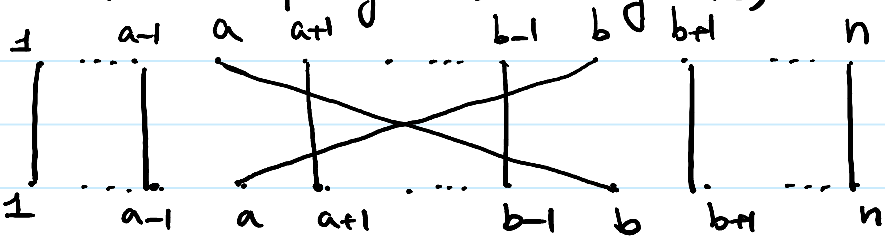
One can use "box of signs" to understand n_σ as well:



permute rows & columns according to σ $\Big\} \rightsquigarrow$ n_σ is # of $-$ that go to the upper half.

<u>Lemma</u> $n_{(a\ b)} = 2(b-a)-1$.

<u>Pf</u> In class we used "box of signs" to study this; here I use crossing #:



$2(b-a-1)+1$ ∎

Lecture 09: Parity

Friday, October 26, 2018 1:18 AM

<u>Theorem.</u> (a) Suppose $\tau_1, \dots, \tau_m$ and $\sigma_1, \dots, \sigma_n$ are transpositions and $\tau_1 \cdot \cdots \cdot \tau_m = \sigma_1 \cdot \cdots \cdot \sigma_n$; then $m \equiv n \pmod 2$.

(b) $\ker \epsilon = \{\sigma \in S_n \mid \sigma$ can be written as a product of even # of transpositions$\}$.

<u>Pf.</u> (a) $\tau_1 \cdot \cdots \cdot \tau_m = \sigma_1 \cdot \cdots \cdot \sigma_n \Rightarrow \epsilon(\tau_1 \cdots \tau_m) = \epsilon(\sigma_1 \cdots \sigma_n)$ (*)

By the previous lemma, $\epsilon(a\ b) = (-1)^{2(b-a)-1} = -1$. And so (*) implies $(-1)^m = (-1)^n$; therefore $m \equiv n \pmod 2$.

(b) $\sigma \in \ker \epsilon$ can be written as a product of transpositions $\tau_1, \dots, \tau_m$. So $1 = \epsilon(\sigma) = \epsilon(\tau_1 \cdot \cdots \cdot \tau_m) = (-1)^m$; and so m is even. By a similar argumen LHS $\supseteq$ RHS. ∎

A nice application of parity of permutations is on possible patterns in a 15-puzzle. In a 15-puzzle, there are 15 small squares (numbered) and an empty spot.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

And one can use the empty spot to move around numbers. $\boxed{Q}$ Can one reach to the following pattern

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

? The answer is NO! Let's number the empty spot 16; and think about any possible pattern as an element of S_{16}. Notice any move $\uparrow, \downarrow, \leftarrow,$ and $\rightarrow$ can be viewed as a transposition that involves 16. If we reach to the new pattern, <u>16</u> is moved back to its initial position. So $\#\uparrow$'s $= \#\downarrow$'s and $\#\rightarrow$'s $= \#\leftarrow$'s. Therefore it should be an even permutation of S_{16}; so it cannot be $(14\ 15)$. $\blacksquare$

<u>Def</u>. $\ker \epsilon$ is called <u>Alternating group</u>; and it is denoted by A_n. Elements of A_n are called even and elements of $S_n \setminus A_n$ are called odd.

<u>Observe</u>. $A_n \trianglelefteq S_n$ and $S_n/A_n \simeq \{\pm 1\}$ if $n \geq 2$.

Next we will show that A_n is simple if $n \geq 5$. We start with a few lemmas.

Lecture 09: 3-cycles and A_n

Friday, October 26, 2018 1:32 AM

Lemma 1. A_n is generated by 3-cycles.

Pf. Since any element of A_n is a product of even number of transpositions, it is enough to write a product of two transp. as a product of 3-cycles. Now notice:

$(a\ b)(a\ b) = I$; $(a\ b)(b\ c) = (a\ b\ c)$; linking

$(a\ b)(c\ d) = (a\ b)(b\ c)(b\ c)(c\ d) = (a\ b\ c)(b\ c\ d)$;

And $(a\ b\ c) = (a\ b)(b\ c) \in A_n$. ∎

Lemma 2 $N \triangleleft A_n$, $n \geq 5$, and N contains a 3-cycle. Then

$$N = A_n.$$

Pf. • Suppose τ_1 and τ_2 are two 3-cycles. So $\exists\, \sigma \in S_n$ s.t. $\sigma \tau_1 \sigma^{-1} = \tau_2$. Since $|\mathrm{Supp}\, \tau_1| = 3$ and $n \geq 5$, $\exists\, a \neq b$ s.t. $\{a, b\} \cap \mathrm{Supp}\, \tau_1 = \emptyset$. Hence $(a\ b)\, \tau_1\, (a\ b) = \tau_1$; and so $(\sigma (a\ b))\, \tau_1\, (\sigma\, (a\ b))^{-1} = \tau_2$. Either $\sigma \in A_n$ or $\sigma (a\ b) \in A_n$, which implies τ_2 is a conjugate of τ_1 in A_n.

• Since $N \triangleleft A_n$ and it contains a 3-cycle τ, all the

Lecture 09: A_5 is simple

Friday, October 26, 2018 9:33 AM

conjugates of τ are in N; and so all 3-cycles are A_n.

And claim follows from Lemm 1. ∎

<u>Lemma 3.</u> A_5 is simple; that means there is no $\{I\}\neq N \trianglelefteq_{\neq} A_5$.

<u>Pf.</u> Suppose to the contrary that $\exists\ \{I\}\neq N \trianglelefteq_{\neq} A_5$.

<u>Case 1.</u> $3 \mid |N|$.

In this case N has an element of order 3.

$$\boxed{\begin{array}{l}\text{The cycle type of an element of order } p \text{ is} \\ p\geq p\geq \cdots \geq p \geq 1 \geq \cdots \geq 1.\end{array}}$$

So the only possible cycle type of this element is $3\geq 1 \geq 1$; this means N has a 3-cycle. Therefore by Lemma 2 $N=A_5$ which is a contradiction.

<u>Case 2.</u> $5 \mid |N|$, $3 \nmid |N|$.

In this case N has a subgroup of order 5. Since $5^2 \nmid |A_5|$, N has a Sylow 5-subgroup of A_5 as a subgroup. Therefore all the Sylow 5-subgps of A_5 are subgps of N. Hence all

Lecture 09: A_5 is simple

Friday, October 26, 2018 9:45 AM

elements of order 5 are in N. The cycle type of an element of order 5 in S_5 is 5; that means it is a 5-cycle. Number of 5-cycles is $4! = 24$; and so $|N| \geq 25$. As $|N| \mid 60$, we deduce $|N| = 30$; this implies $3 \mid |N|$ which is a contradiction.

<u>Case 3.</u> $\left.\begin{array}{l} 3 \nmid |N|, 5 \nmid |N| \\ |N| \mid 60 \end{array}\right\} \Rightarrow |N| \mid 4.$

N has an element of order 2. Possible cycle types of an element of order 2 are $2 \geq 1 \geq 1 \geq 1$ and $2 \geq 2 \geq 1$. Since the first type gives us an <u>odd</u> permutation and $N \subseteq A_5$, $\exists$ an element of the form $(a\ b)(c\ d)$ in N. After reordering we can assume $\underbrace{(1\ 2)(3\ 4)}_{\sigma} \in N$; and so

$(1\ 2\ 3)\ \sigma\ (1\ 2\ 3)^{-1} = (2\ 3)(1\ 4) \in N$,

$(1\ 3\ 2)\ \sigma\ (1\ 3\ 2)^{-1} = (3\ 1)(2\ 4) \in N$,

$(1\ 5)(3\ 4)\ \sigma\ (1\ 5)(3\ 4) = (5\ 2)(4\ 3) \in N$.

$\Rightarrow |N| > 4$ which is a contradiction. ∎

Lecture 09: A_n is simple

Friday, October 26, 2018 10:02 AM

<u>Theorem</u>. A_n is simple if $n \geq 5$.

<u>Pf.</u> We proceed by induction on n. We have already proved the base of induction. Next we will prove the induction step for $n \geq 6$. Let $G_i := \{\sigma \in A_n \mid \sigma(i) = i\}$. Notice that

$$G_i \rightleftarrows A_{[1..n] \setminus \{i\}} \qquad \sigma \mapsto \bar{\sigma} \text{ where } \bar{\sigma}(j) = \sigma(j).$$

$$\bar{\sigma} \mapsto \sigma \text{ where } \sigma(i) = i \text{ and } \sigma(j) = \bar{\sigma}(j) \text{ if } j \neq i$$

are group homomorphisms that are inverse of each other.

So $G_i \simeq A_{n-1}$ for any $1 \leq i \leq n$. So by the induction hypothesis G_i's are simple. Suppose $I \neq N \underset{\neq}{\trianglelefteq} A_n$. Then $N \cap G_i \trianglelefteq G_i$ for any i. Since G_i is simple, either $G_i \cap N = G_i$ or $G_i \cap N = \{I\}$. If $G_i \cap N = G_i$, then N contains a 3-cycle. Lemma 2 implies $N = A_n$ which is a contradiction. So for any i, $G_i \cap N = I$; that means $\forall\, \underset{\neq I}{\bar{\sigma}} \in N, \forall i, \bar{\sigma}(i) \neq i$. Hence

$$\forall\, \sigma \neq \tau \in N, \forall i, \sigma(i) \neq \tau(i). \quad (*)$$

Next we will find two elements of N that violate $(*)$.

Lecture 09: A_n is simple

Friday, October 26, 2018 1:16 PM

Suppose $\sigma \in N \setminus I$ and $p_1 \geq p_2 \geq \cdots$ is its cycle type.

Case 1 $p_1 \geq 3$.

So $\sigma = \underbrace{(a\ b\ c\ \cdots)}_{p_1}\underbrace{(\cdots)}_{p_2}\cdots \in N$. Since $n \geq 6$, $\exists e, f$,

$\{e,f\} \cap \{a,b,c\} = \emptyset$. And so

$$\underbrace{(c\ e\ f)\ \sigma\ (c\ e\ f)^{-1}}_{\sigma'} = (a\ b\ e\ \cdots)(\cdots)\cdots \in N$$

and $\sigma(a) = \sigma'(a) = b$, which contradicts $(*)$.
$\sigma \neq \sigma'$ as $\sigma(b) \neq \sigma'(b)$

Details of case 2 were not discussed in class

Case 2. $p_1 = 2$

Since $\sigma \in A_n$, $p_2 = 2$; and so $\sigma = (a\ b)(c\ d)(\cdots)\cdots$

Since $n \geq 6$, $\exists e, f$, $\{e,f\} \cap \{a,b,c,d\} = \emptyset$; and so

$$\underbrace{(d\ e\ f)\ \sigma\ (d\ e\ f)^{-1}}_{\sigma'} = (a\ b)(c\ e)(\cdots) \in N$$

and $\sigma(a) = \sigma'(a) = b$ and $\sigma \neq \sigma'$ as $\sigma(c) \neq \sigma'(c)$, which contradicts $(*)$. ■