Lecture 11: Solvable groups

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Def. A group G is called solvable if $\exists k \in \mathbb{Z}^{\geq 0}$, $G^{(k)} = 1$.

(Name is given because of a theorem by Galois on solvability of a polynomial by radicals.)

Lemma. $\phi: G \to H$ a group homomorphism $\Rightarrow$
$\forall i$, $\phi(G^{(i)}) = \phi(G)^{(i)}$.

Pf. Exercise, prove this by induction on i.

Theorem. Suppose G is a solvable group, $H \leq G$, and $N \trianglelefteq G$.

Then (1) H is solvable, (2) G/N is solvable.

Pf. (1) By induction on i, $\left.\begin{array}{l} H^{(i)} \subseteq G^{(i)} \\ G^{(k)} = 1 \end{array}\right\} \Rightarrow H^{(k)} = 1$,

(2) $\pi: G \twoheadrightarrow G/N \Rightarrow \pi(G)^{(k)} = \pi(G^{(k)}) = 1 \Rightarrow (G/N)^{(k)} = 1$. ∎

Proposition. Suppose G is a finite group. Then

G is solvable $\Leftrightarrow$ all the composition factors of G are cyclic groups of prime order.

Pf. ($\Rightarrow$) Suppose $1 = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_k = G$ is a composition series of G. Then N_i's are solvable; and so N_i/N_{i-1}'s are solvable. Therefore the following claim implies this direc.

Lecture 11: Composition factors of solvable groups

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__Claim__. A solvable simple group is a cyclic group of prime order.

__Pf of Claim.__ $\left.\begin{array}{l} H^{(k)}=1 \\ H\neq 1\end{array}\right\} \Rightarrow \left.\begin{array}{l} H^{(1)}\neq H \\ H^{(1)}\lhd H \\ H \text{ simple}\end{array}\right\} \Rightarrow H^{(1)}=1 \Rightarrow H$ abelian

$\Rightarrow \left.\begin{array}{l}\text{any subgp is normal} \\ H \text{ is simple}\end{array}\right\} \Rightarrow H$ is a cyclic gp of prime order. $\square$

$\Leftarrow$) Suppose $1=N_0 \lhd \cdots \lhd N_k = G$ is a composition series and N_i/N_{i-1} is cyclic. Then by a lemma that we proved earlier $G^{(k)}=1$; and so G is solvable. $\blacksquare$

__Def.__ • Let $\gamma_1(G) := G$, $\gamma_{i+1}(G) := [\gamma_i(G), G]$. $\{\gamma_i(G)\}_{i=1}^{\infty}$ is called the __lower central series__ of G.

• Let $Z_0(G) := \{1\}$, $Z\left(G/Z_i(G)\right) =: Z_{i+1}(G)/Z_i(G)$.

$\{Z_i(G)\}_{i=0}^{\infty}$ is called __the upper central series__ of G.

__Basic Properties__ (1) $\gamma_i(G)$ is a char. subgroup

(2) $\gamma_1(G) \trianglerighteq \gamma_2(G) \trianglerighteq \cdots$.

(3) $Z\left(G/\gamma_{i+1}(G)\right) \supseteq \gamma_i(G)/\gamma_{i+1}(G)$.

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(4) $Z_i(G)$ is a char. subgp.

(5) $Z_{i+1}(G)/Z_i(G)$ is abelian.

<u>Pf.</u> (1) We proceed by induction on i; base case is clear.

$\forall \theta \in \mathrm{Aut}(G)$, $\theta(\gamma_{i+1}(G)) = \theta([\gamma_i(G), G]) = [\theta(\gamma_i(G)), \theta(G)]$

$$= [\gamma_i(G), G] = \gamma_{i+1}(G).$$

(2) $[\gamma_i(G), G] \subseteq \gamma_i(G)$ as $\gamma_i(G) \trianglelefteq G$.

(3) $\forall g \in G$, $g' \in \gamma_i(G)$, $[g, g'] \in \gamma_{i+1}(G)$; and so

$(g\,\gamma_{i+1}(G))(g'\,\gamma_{i+1}(G)) = (g'\,\gamma_{i+1}(G))(g\,\gamma_{i+1}(G))$, which

implies $g'\,\gamma_{i+1}(G) \in Z(G/\gamma_{i+1}(G))$.

(4) We proceed by induction on i; the base case is clear.

$\forall \theta \in \mathrm{Aut}(G)$, since $\theta(Z_i(G)) = Z_i(G)$,

$\bar{\theta}: G/Z_i(G) \to G/Z_i(G)$, $\bar{\theta}(g Z_i(G)) := \theta(g) Z_i(G)$ is

a well-defined automorphism; and so $\bar{\theta}(Z(G/Z_i(G))) = Z(G/Z_i(G))$

This implies $\theta(Z_{i+1}(G)) \underbrace{Z_i(G)}_{= \theta(Z_i(G))} = Z_{i+1}(G) \Rightarrow \theta(Z_{i+1}(G)) = Z_{i+1}(G)$

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(5) $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$; and so it is abelian. ∎

<u>Theorem</u>. For a non-negative integer c,

$$\gamma_{c+1}(G) = \{1\} \Leftrightarrow Z_c(G) = G.$$

<u>Pf.</u> ($\Rightarrow$) We prove by induction on i that

$$\gamma_{c+1-i}(G) \subseteq Z_i(G).$$

<u>Base</u> follows from $\gamma_{c+1}(G) = \{1\}$.

<u>Induction Step.</u> To show $\gamma_{c-i}(G) \subseteq Z_{i+1}(G)$, one has to show $\forall g' \in \gamma_{c-i}(G)$, $g' Z_i(G) \in Z(G/Z_i(G))$; that means $\forall g \in G$, $[g, g']$ should be in $Z_i(G)$ (?)

$[g, g'] \in [G, \gamma_{c-i}(G)] = \gamma_{c-i+1}(G) \subseteq Z_i(G)$, and claim follows. And so $G = \gamma_1(G) \subseteq Z_c(G)$.

($\Leftarrow$) By induction on i, we prove $\gamma_i(G) \subseteq Z_{c+1-i}(G)$.

Since $Z_c(G) = G$, the base case follows.

To prove the induction step, we have to show $\gamma_{i+1}(G) \subseteq Z_{c-i}(G)$.

Notice that $Z_{c-i+1}(G)/Z_{c-i}(G) = Z(G/Z_{c-i}(G))$.

$$\left.\begin{aligned} \Rightarrow [G, Z_{c-i+1}(G)] &\subseteq Z_{c-i}(G) \\ \gamma_i(G) &\subseteq Z_{c-i+1}(G) \end{aligned}\right\} \Rightarrow [G, \gamma_i(G)] \subseteq Z_{c-i}(G) \Rightarrow \gamma_{i+1}(G) \subseteq Z_{c-i}(G);$$

and claim follows. In particular, $\gamma_{c+1}(G) \subseteq Z_0(G) = \{1\}$. ■

<u>Def.</u> A group G is called <u>nilpotent</u> if $\exists\, c \in \mathbb{Z}^{\geq 0}$ s.t.

$$\gamma_{1+c}(G) = \{1\}.$$

(Alternatively $Z_c(G) = G$). In this case we say the nilpotency class of G is c.

<u>Proposition</u>. A finite p-group is nilpotent.

<u>Pf.</u> We proceed by strong induction on $|G|$. If $|G| = 1$, we are done. If not, $Z(G) \neq 1$. And so $|G/Z(G)| < |G|$ and $G/Z(G)$ is a finite p-group. So $\exists\, c \in \mathbb{Z}^+$ s.t. the c^{th} group in the upper central series of $G/Z(G)$ is $G/Z(G)$. Notice that the upper central series of $G/Z(G)$ is $\{Z_i(G)/Z(G)\}_{i=1}^{\infty}$; and so $Z_c(G) = G$. ■

Lecture 11: Nilpotent groups

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<u>Proposition</u>. Suppose G is a nilpotent group. Then

$$H \lneq G \Rightarrow H \lneq N_G(H).$$

<u>Pf.</u> Since G is nilpotent, $\exists c \in \mathbb{Z}^{\geq 0}$, $Z_c(G) = G$. Let

$i_0 < c$ be s.t. $Z_{i_0}(G) \subseteq H$ and $Z_{i_0+1}(G) \not\subseteq H$.

Let $g \in Z_{i_0+1}(G) \setminus H \Rightarrow \begin{cases} g Z_{i_0}(G) \in Z(G/Z_{i_0}(G)) \\ g \notin H \end{cases}$

$\Rightarrow [g, H] \subseteq Z_{i_0}(G) \subseteq H \Rightarrow \forall h \in H,\ g^{-1} h^{-1} g h \in H$

$\Rightarrow g^{-1} h^{-1} g \in H \Rightarrow g^{-1} H g \subseteq H$

Similarly $g H g^{-1} \subseteq H \Leftarrow\!\Rightarrow g \in N_G(H) \setminus H$. ∎

<u>Corollary</u>. Suppose G is a finite nilpotent group. Then any Sylow p-subgp of G is normal.

<u>Pf.</u> We know $N_G(N_G(P)) = N_G(P)$ if P is a Sylow p-subgp.

Hence, by the above proposition, $N_G(P) = G$; this means

$P \trianglelefteq G$. ∎

Lecture 11: Characterization of finite nilpotents groups

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<u>Theorem</u>. Suppose G is a finite group. Then the following statements are equivalent:

(1) G is nilpotent.

(2) $\forall p \mid |G|$, G has a unique Sylow p-subgroup.

(3) $G \simeq \prod_{i=1}^{m} P_i$ where P_i is a finite p_i-group.

<u>Pf</u>. (1) $\Rightarrow$ (2) is proved in the previous corollary.

(2) $\Rightarrow$ (3) Suppose $|G| = \prod_{i=1}^{m} p_i^{k_i}$ where $k_i \in \mathbb{Z}^+$ and p_i's are distinct prime numbers. Let P_i be the unique Sylow p_i-subgroup of G. So $P_i \trianglelefteq G$; and since $\gcd(|P_i|, |P_j|) = 1$ for $i \neq j$, P_i and P_j commute. In particular, for any ℓ, $P_1 \cdot P_2 \cdots P_\ell$ is a (normal) subgroup of G.

<u>Claim</u>.
$$P_1 \times P_2 \times \cdots \times P_\ell \xrightarrow{\Phi_\ell} P_1 \cdot P_2 \cdots P_\ell, \quad (g_1, g_2, \ldots, g_\ell) \longmapsto g_1 \cdot g_2 \cdots g_\ell$$
is an isomorphism.

<u>Pf of Claim</u>. We proceed by induction ℓ. The <u>base</u> case is clear.

Lecture 11: Characterization of finite nilpotent groups

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<u>Induction Step</u>:

$$P_1 \times \cdots \times P_\ell \times P_{\ell+1} \xrightarrow[(\Phi_\ell,\, id.)]{\sim} P_1 \cdot P_2 \cdots P_\ell \times P_{\ell+1} \xrightarrow{\Phi} P_1 \cdots P_{\ell+1}$$

$$(g\,,\,g') \longmapsto gg'$$

$$\xrightarrow[\Phi_{\ell+1}]{}$$

By the induction hypothesis, $(\Phi_\ell, id.)$ is an isomorphism (and $|P_1 \cdots P_\ell| = \prod_{i=1}^{\ell} |P_i|$) So it is enough to show the 2^{nd} map is a group homomorphism.

$$\left.\begin{array}{l} P_1 \cdots P_\ell \triangleleft G \\ P_{\ell+1} \triangleleft G \\ \gcd(|P_1 \cdots P_\ell|, |P_{\ell+1}|) = 1 \end{array}\right\} \Rightarrow P_1 \cdots P_\ell \cap P_{\ell+1} = 1, \text{ and } P_1 \cdots P_\ell \text{ and } P_{\ell+1} \text{ commute.}$$

Based on these it is easy to see why Φ is an isomorphism (why?)

(3) $\Rightarrow$ (1) We have already proved that P_i's are nilpotent. So $\exists\, c \in \mathbb{Z}^+$ s.t. $\gamma_c(P_i) = \{1\}$ for $1 \le i \le m$. Then

$$\gamma_c(G) \simeq \gamma_c(\prod_i P_i) \underset{(?)}{=} \prod_i \gamma_c(P_i) = \{1\}. \qquad \blacksquare$$

(Justify this)

Let me finish today's lecture by another characterization of finite nilpotent groups:

<u>Theorem</u>. A finite group is nilpotent if and only if all maximal

subgroups are normal.

<u>Pf.</u> ($\Rightarrow$) $M \lneq G \Rightarrow \left.\begin{array}{l} M \lneq N_G(M) \\ M \text{ is maximal} \end{array}\right\} \Rightarrow N_G(M) = G \Rightarrow M \trianglelefteq G$.

($\Leftarrow$) Let P be a Sylow p-subgp. We would like to show $P \trianglelefteq G$. Suppose to the contrary $N_G(P) \lneq G$. Then there is a maximal subgroup M of G s.t. $N_G(P) \leq M$ (since G is finite, there is such a subgroup.) And so $M \trianglelefteq G$ and P is a Sylow p-subgroup of M. Hence by Fratini's argument $G = N_G(P) \cdot M \subseteq M$ which is a contradiction. ∎