Lecture 14: Ping-pong lemma

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In the previous lecture we defined free group and proved its universal property. How can we prove if a given group is free or not?

<u>Ping-pong lemma</u>. Suppose $G \curvearrowright X$, $G_1, G_2 \leq G$, $|G_1| \geq 2$, $|G_3| \geq 3$; let $X_1, X_2 \subseteq X$, $X_1 \nsubseteq X_2$ and $X_2 \nsubseteq X_1$. Suppose $(G_1 \setminus \{1\}) \cdot X_2 \subseteq X_1$ and $(G_2 \setminus \{1\}) \cdot X_1 \subseteq X_2$. Then

$$\langle G_1 \cup G_2 \rangle \simeq G_1 * G_2 .$$

<u>Pf.</u> Let $\phi_1 : G_1 \hookrightarrow \langle G_1 \cup G_2 \rangle$ and $\phi_2 : G_2 \hookrightarrow \langle G_1 \cup G_2 \rangle$. Then by the universal property of free prod. $\exists\ \phi : G_1 * G_2 \to \langle G_1 \cup G_2 \rangle$ s.t. $\phi|_{G_i} = \phi_i$; in particular ϕ is onto.

Suppose $\omega \in \ker \phi \subseteq G_1 * G_2$. We consider the unique <u>reduced</u> form of ω:

<u>Case 1.</u> $\omega = a_1 b_1 a_2 b_2 \cdots a_n b_n a_{n+1}$, $a_i \in G_1 \setminus 1$, $b_i \in G_2 \setminus 1$.

Suppose $x_2 \in X_2 \setminus X_1$. Then

$$x_2 = \phi(\omega) \cdot x_2 = a_1 \cdot (b_1 \cdot \cdots \cdot (a_n \cdot \overbrace{(b_n \cdot \underbrace{(a_{n+1} \cdot x_2)}_{\in X_1})}^{\in X_2})) \in X_1$$

which is a contradiction.

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Case 2. $\omega = b_1 a_1 b_2 a_2 \cdots\cdots b_n a_n b_{n+1}$, $a_i \in G_1 \setminus 1$, $b_i \in G_2 \setminus 1$.

Suppose $x_1 \in X_1 \setminus X_2$. Then

$$x_1 = \phi(\omega)\cdot x_1 = \underbrace{b_1 \cdot a_1 \cdot b_2 \cdot \cdots \underbrace{a_n \cdot \underbrace{b_{n+1} \cdot x_1}_{\text{in } X_2}}_{\text{in } X_1}}_{\text{in } X_2} \in X_2$$

which is a contradiction.

Case 3. $\omega = a_1 b_1 a_2 b_2 \cdots a_n b_n$, $a_i \in G_1 \setminus 1$, $b_i \in G_2 \setminus 1$.

Since $|G_2| \geq 3$, $\exists\, b \in G_2 \setminus \{1, b_n^{-1}\}$; then

$b\omega b^{-1} = b a_1 b_1 a_2 b_2 \cdots a_n \underbrace{(b_n b^{-1})}_{\text{in } G_2 \setminus 1}$ is reduced and $b\omega b^{-1} \in \ker \phi$.

we get a contrad. by case 2.

Case 4. $\omega = b_1 a_1 b_2 a_2 \cdots b_n a_n$, $b_i \in G_2 \setminus 1$, $a_i \in G_1 \setminus 1$.

$\Rightarrow$ $b_1^{-1} \omega b_1 = a_1 b_2 \cdots a_{n-1} b_n a_n b_1$ is reduced and in $\ker \phi$

we get a contradiction by case 3. ∎

Ex. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ freely generate a subgroup of $SL_2(\mathbb{Z})$.

Pf. Let $G_1 := \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix} \,\middle|\, n \in \mathbb{Z} \right\}$ and

$$G_2 := \left\langle \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix} \,\middle|\, n \in \mathbb{Z} \right\}.$$

Lecture 14: Applications of ping-pong lemma

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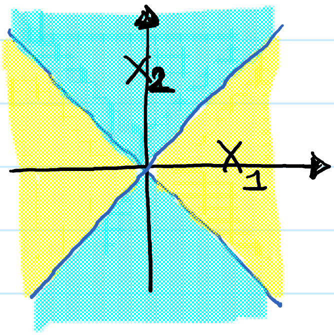
$SL_2(\mathbb{Z}) \curvearrowright \mathbb{P}(\mathbb{R}^2)$ projective space



$X_1 := \{[x:y] \mid |y| \le |x|\}$,

$X_2 := \{[x:y] \mid |y| \ge |x|\}$,

Claim. $(G_1 \setminus 1) \cdot X_2 \subseteq X_1$.

Pf of Claim. $\begin{bmatrix} 1 & 2n \\ & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2ny \\ y \end{bmatrix}$

$$|x+2ny| \ge |2n||y| - |x| \ge |y| + (|y|-|x|)$$
$$\ge |y|.$$

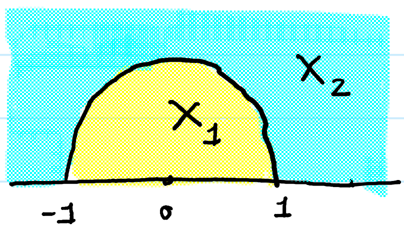
Similarly $(G_2 \setminus 1) \cdot X_1 \subseteq X_2$. So, by Ping-pong lemma,

$$\left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\rangle \simeq G_1 * G_2 \simeq \mathbb{Z} * \mathbb{Z} = F_2. \quad \blacksquare$$

Ex. $\left\langle \overline{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}, \overline{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \right\rangle \simeq \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ where $\bar{g} \in PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\pm I$ for $g \in SL_2(\mathbb{R})$.

Pf. $SL_2(\mathbb{R}) \curvearrowright \mathcal{H}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \dfrac{az+b}{cz+d}$ as you have seen in one of your HW assignments. Notice that this action factors through $PSL_2(\mathbb{R})$. Let $G_1 := \left\langle \overline{\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}} \right\rangle$ and

$G_2 := \left\langle \overline{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}} \right\rangle$.



Lecture 14: Applications of ping-pong lemma

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Notice that $\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\cdot z = \frac{-1}{z}$ and $\begin{bmatrix} 1 & 2n \\ & 1 \end{bmatrix}\cdot z = z+2n$;

and so $(G_1\setminus 1)\cdot X_2 \subseteq X_1$ and $(G_2\setminus 1)\cdot X_1 \subseteq X_2$. Therefore

by ping-pong lemma, $\langle \overline{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}, \overline{\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}} \rangle \simeq \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. ■

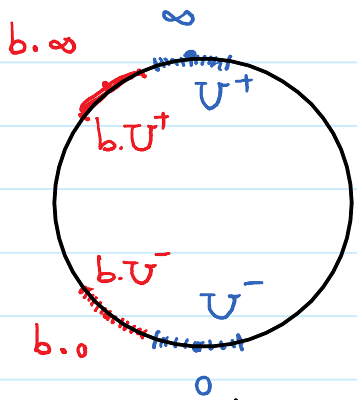
<u>Ex.</u> (Schottky group) Let $a = \begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}$, $\lambda > 1$, and $b \in SL_2(\mathbb{R})$

s.t. $b\bullet\{0,\infty\} \cap \{0,\infty\} = \emptyset$. Then, for large enough n,

a^n and $b a^n b^{-1}$ freely generate a subgroup of $SL_2(\mathbb{R})$.

<u>Pf.</u> $SL_2(\mathbb{R}) \curvearrowright \underbrace{\mathbb{R}\cup\{\infty\}}_{S}$ by Möbius transformation. Let's

use the circle model of $\mathbb{R}\cup\{\infty\}$;



$a^m \cdot z = \lambda^{2m} z$ so a is contracting

$S\setminus 0$ to ∞ and a^{-1} is contracting

$S\setminus\infty$ to 0. So $\exists$ a nbhd $U^- \ni 0$ and a nbhd $U^+ \ni \infty$

s.t. $a^n\cdot(S\setminus U^-) \subseteq U^+$ for $n \geq n_0$ and

. $a^{-n}(S\setminus U^+) \subseteq U^-$ for $n \geq n_0$.

. $b.U^{\pm} \cap U^{\pm} = \emptyset$; (Since $b.\{0,\infty\}\cap\{0,\infty\} = \emptyset$, there are $U^{\pm}$.)

Let $G_1 := \langle a^{n_0} \rangle$, $G_2 := \langle b a^{n_0} b^{-1} \rangle$, $X_1 := U^- \cup U^+$, $X_2 = b.X_1$.

Lecture 14: Schottky groups

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Then $a^{n_0 k}.(bU^+ \cup bU^-) \subseteq a^{n_0 k}(S \setminus (U^+ \cup U^-))$

$$\subseteq U^+ \cup U^-$$

$$(b a^{n_0 k} b^{-1})(U^+ \cup U^-) \subseteq b a^{n_0 k}(b^{-1}U^+ \cup b^{-1}U^-)$$

$$\subseteq b a^{n_0 k}(S \setminus (U^+ \cup U^-))$$

$$\subseteq b(U^+ \cup U^-);$$ and so by

ping-pong lemma $\langle a^{n_0}, b a^{n_0} b^{-1} \rangle \simeq \mathbb{Z} * \mathbb{Z} = F_2$. ■

<u>Theorem</u>. Let $a = \begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}$, $\lambda > 1$, $b \in SL_2(\mathbb{R}) \setminus \left(\left\{ \begin{bmatrix} * & * \\ & * \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} * & \\ * & * \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} & * \\ * & \end{bmatrix} \right\} \right)$.

Then $\langle a, b \rangle$ has a non-commutative free subgp.

J. Tits proved the generalization of the above theorem based on action on projective space.

<u>Theorem</u>. Suppose $\Gamma \leq GL_n(\mathbb{C})$ is a finitely generated linear group, which is not <u>virtually</u> solvable; that means no subgp of finite index of Γ is solvable. Then Γ has a (non-commut.) free subgp.

(In your HW assignment you will show its inverse.)

Lecture 14: Presentation

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<u>Def.</u> Suppose $R \subseteq F(X)$. Then $\langle X \mid R \rangle$ means $F(X)/N$

where $N = \langle \bigcup_{g \in F(X)} gRg^{-1} \rangle$ (is the smallest normal subgp

of $F(X)$ that contains R).

In general it is not easy to understand the group structure of a group with a given presentation; to be more precise for a given presentation $\langle X \mid R \rangle$ and a given word $\omega \in F(X)$ one can ask if $\omega = e$ in $\langle X \mid R \rangle$. Is there an algorithm to check whether $\omega = e$? This is called the word problem, and Novikov showed that in general answer to this question is NO.

In certain cases we can understand group structure of $\langle X \mid R \rangle$. Next we describe a general strategy, and start with the following easy example:

<u>Ex.</u> $\langle a \mid a^n \rangle \simeq \mathbb{Z}/n\mathbb{Z}$.

(We'll continue in the next lecture.)