Lecture 17: Prime and maximal ideals

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In the previous lecture we defined prime and maximal ideals.

The following lemma gives us the connection between these proper. of an ideal and the corresponding factor ring.

<u>Lemma</u>. Suppose A is a unital commutative ring and $I \trianglelefteq A$.

(1) I is prime $\Leftrightarrow A/I$ is an integral domain.

(2) I is maximal $\Leftrightarrow A/I$ is a field.

Pf. (1) ($\Rightarrow$) Since I is prime, it is proper. And so A/I is not the trivial ring.

$$(a+I)(b+I) = 0+I \Rightarrow ab+I = 0+I \Rightarrow ab \in I$$
$$\Rightarrow a\in I \text{ or } b \in I$$
$$\Rightarrow a+I = 0 \quad \text{or} \quad b+I=0$$

So A/I has no zero-divisor.

($\Leftarrow$). A/I is an integral domain $\Rightarrow A/I$ is not the trivial ring

$\Rightarrow I \neq A$.

- $ab \in I \Rightarrow (a+I)(b+I) = 0 \Rightarrow a+I=0 \quad \text{or} \quad b+I=0$

$\Rightarrow a \in I$ or $b \in I$.

(2) ($\Rightarrow$) Since I is maximal, it is a proper ideal; and so A/I is

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not the trivial ring. Suppose $a+I \neq 0$. Then $a \notin I$. So

$I \subsetneq \langle a \rangle + I \trianglelefteq A$. By maximality of I, we deduce that

$\langle a \rangle + I = A$; and so $\exists\, b \in A$ and $c \in I$ s.t. $ab+c=1$.

This implies $(a+I)(b+I) = 1+I$; and that means $a+I$ is

a unit; and so A/I is a field.

($\Leftarrow$) Since A/I is a field, it is a non-trivial ring. And so I is

a proper ideal. Suppose $I \subsetneq J \trianglelefteq A$. Let $a \in J \setminus I$. Then

$a+I$ is a unit in A/I. So $\exists\, b \in A$ s.t. $(a+I)(b+I) = 1+I$.

Therefore $\exists\, c \in I$ s.t. $ab+c=1$, which implies

$$1 = ab+c \in \langle a \rangle + I \subseteq J;$$ and so $J = A$. Thus I is

a maximal ideal. ∎

<u>Corollary</u>. $\mathrm{Max}(A) \subseteq \mathrm{Spec}(A)$.

<u>Pf</u>. $\mathfrak{m} \in \mathrm{Max}(A) \Rightarrow A/\mathfrak{m}$ is a field $\Rightarrow A/\mathfrak{m}$ is an integral domain

$\Rightarrow \mathfrak{m} \in \mathrm{Spec}(A)$. ∎

<u>Cor</u>. $\mathfrak{p} \in \mathrm{Spec}(A)$ and $|A/\mathfrak{p}| < \infty \Rightarrow \mathfrak{p} \in \mathrm{Max}(A)$.

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Pf. $\mathfrak{p} \in \mathrm{Spec}(A) \Rightarrow \left.\begin{array}{l} A/\mathfrak{p} \text{ is an integral domain} \\ |A/\mathfrak{p}| < \infty \end{array}\right\} \Rightarrow$

$A/\mathfrak{p}$ is a field $\Rightarrow \mathfrak{p} \in \mathrm{Max}(A)$. ■

Next we will find "lots" of prime and maximal ideals. To show this we use Zorn's lemma. This result is equivalent to axiom of choice.

Def. A non-empty set Σ with a relation $\preccurlyeq$ is called a Partially Ordered Set (POSet) if

- $a \preccurlyeq a$
- $\left.\begin{array}{l} a \preccurlyeq b \\ b \preccurlyeq a \end{array}\right\} \Rightarrow a = b$
- $\left.\begin{array}{l} a \preccurlyeq b \\ b \preccurlyeq c \end{array}\right\} \Rightarrow a \preccurlyeq c$.

Def. Suppose $(\Sigma, \preccurlyeq)$ is a poset. A non-empty subset C of Σ is called a chain if $\forall\, a, b \in C$, either $a \preccurlyeq b$ or $b \preccurlyeq a$.

(It is also called a totally ordered set.)

Def. Suppose $(\Sigma, \preccurlyeq)$ is a poset, and Δ is a non-empty subset of Σ. We say $a \in \Sigma$ is an upper bound of Δ if $\forall b \in \Delta$, $b \preccurlyeq a$.

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Def. Suppose $(\Sigma, \preccurlyeq)$ is a poset, $a \in \Sigma$ is called a maximal element of Σ if $a \preccurlyeq b$ implies $a = b$.

Zorn's lemma. Suppose $(\Sigma, \preccurlyeq)$ is a poset. If any chain $C \subseteq \Sigma$ has an upper bound, then Σ has a maximal element.

- Proof of the next theorem is a good example of how Zorn's lemma can be used in algebra.

Def. Suppose A is a unital commutative ring. A subset $S \subseteq A$ is called multiplicatively closed if $1 \in S$ and

$s_1, s_2 \in S \Rightarrow s_1 s_2 \in S$.

Theorem. Suppose A is a unital commutative ring, $S \subseteq A$ is multiplicatively closed, $\mathfrak{a} \trianglelefteq A$, and $\mathfrak{a} \cap S = \emptyset$. Let

$$\Sigma_{\mathfrak{a}, S} := \{ \mathfrak{b} \trianglelefteq A \mid \cdot\, \mathfrak{a} \subseteq \mathfrak{b} \quad \cdot\, \mathfrak{b} \cap S = \emptyset \}.$$

(1) $\Sigma_{\mathfrak{a}, S}$ has a maximal element w.r.t. $\subseteq$.

(2) If $\mathfrak{p} \in \Sigma_{\mathfrak{a}, S}$ is a maximal element of $\Sigma_{\mathfrak{a}, S}$, then $\mathfrak{p}$ is prime.

In particular $\mathrm{Spec}(A) \cap \Sigma_{\mathfrak{a}, S} \neq \emptyset$.

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<u>Cor.</u> Suppose $\mathfrak{a} \underset{\neq}{\trianglelefteq} A$. Then $\exists\, \mathfrak{m} \in \operatorname{Max} A$ s.t. $\mathfrak{a} \subseteq \mathfrak{m}$.

<u>Pf. of Cor.</u> Since $\mathfrak{a} \neq A$, $\{1\} \cap \mathfrak{a} = \emptyset$. By part (1) of the previous theorem, $\Sigma_{\mathfrak{a},\{1\}}$ has a maximal element $\mathfrak{m}$.

<u>Claim.</u> $\mathfrak{m} \in \operatorname{Max}(A)$.

<u>Pf of Claim.</u> Suppose $\mathfrak{m} \subsetneq \mathfrak{m}' \underset{\neq}{\trianglelefteq} A$. Then $\mathfrak{a} \subseteq \mathfrak{m} \subseteq \mathfrak{m}'$ and $\mathfrak{m}' \cap \{1\} = \emptyset$. Hence $\mathfrak{m}' \in \Sigma_{\mathfrak{a},\{1\}}$. Since $\mathfrak{m}$ is a maximal element of $\Sigma_{\mathfrak{a},\{1\}}$, we deduce that $\mathfrak{m} = \mathfrak{m}'$. ■

<u>Pf of Theorem</u>. (1) By Zorn's lemma it is enough to show any chain $\mathcal{C} \subseteq \Sigma_{\mathfrak{a},S}$ has an upper bound.

<u>Claim 1.</u> $\bigcup_{\mathfrak{b} \in \mathcal{C}} \mathfrak{b}$ is an ideal of A. (This is true for any chain of ideals; and usually this is how Zorn's lemma is used in ring theory.)

<u>Pf of Claim 1.</u> $a_1, a_2 \in \bigcup_{\mathfrak{b} \in \mathcal{C}} \mathfrak{b} \Rightarrow \exists\, \mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{C}$ s.t. $a_1 \in \mathfrak{b}_1$, $a_2 \in \mathfrak{b}_2$

$\mathcal{C}$ is a chain $\Rightarrow$ either $\mathfrak{b}_1 \subseteq \mathfrak{b}_2$ or $\mathfrak{b}_2 \subseteq \mathfrak{b}_1$ $\Big\} \Rightarrow$

either $a_1, a_2 \in \mathfrak{b}_1$ or $a_1, a_2 \in \mathfrak{b}_2 \Rightarrow a_1 + a_2 \in \mathfrak{b}_1 \cup \mathfrak{b}_2$

$\Rightarrow a_1 + a_2 \in \bigcup_{\mathfrak{b} \in \mathcal{C}} \mathfrak{b}$.

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- $r \in A$, $a \in \bigcup_{\mathfrak{b} \in \mathcal{C}} \mathfrak{b} \Rightarrow \left\{ \begin{matrix} \exists \mathfrak{b} \in \mathcal{C}, \; a \in \mathfrak{b} \\ r \in A \end{matrix} \right\} \Rightarrow ra \in \mathfrak{b}$

$$\Rightarrow ra \in \bigcup_{\mathfrak{b} \in \mathcal{C}} \mathfrak{b}. \qquad \blacksquare \text{ (Claim 1)}$$

<u>Claim 2</u>. If $\mathcal{C} \subseteq \Sigma_{\mathfrak{a},S}$ is a chain, then $\bigcup_{\mathfrak{b} \in \mathcal{C}} \mathfrak{b} \in \Sigma_{\mathfrak{a},S}$.

<u>Pf of Claim 2.</u> . By Claim 1, $\bigcup_{\mathfrak{b} \in \mathcal{C}} \mathfrak{b} \trianglelefteq A$.

- $\bigcup_{\mathfrak{b} \in \mathcal{C}} \mathfrak{b} \supseteq \mathfrak{b}_0 \supseteq \mathfrak{a}$ for any $\mathfrak{b}_0 \in \mathcal{C}$.
- $\left(\bigcup_{\mathfrak{b} \in \mathcal{C}} \mathfrak{b}\right) \cap S = \bigcup_{\mathfrak{b} \in \mathcal{C}} \underbrace{(\mathfrak{b} \cap S)}_{\varnothing} = \varnothing$; and claim follows. $\blacksquare$ (Claim 2)

- By Claim 2, for any chain $\mathcal{C} \subseteq \Sigma_{\mathfrak{a},S}$, $\bigcup_{\mathfrak{b} \in \mathcal{C}} \mathfrak{b}$ is an upper bound of $\mathcal{C}$. And so by Zorn's lemma $\Sigma_{\mathfrak{a},S}$ has a maximal element. $\blacksquare$ (part (1))

(2) Suppose $\mathfrak{p} \in \Sigma_{\mathfrak{a},S}$ is maximal in $\Sigma_{\mathfrak{a},S}$. And suppose to the contrary $\mathfrak{p}$ is not prime. So $\exists\, a, b \in A$ s.t. $a, b \notin \mathfrak{p}$ and $ab \in \mathfrak{p}$. Hence $\mathfrak{p} \subsetneq \mathfrak{p} + \langle a \rangle$ and $\mathfrak{p} \subsetneq \mathfrak{p} + \langle b \rangle$. By maximality of $\mathfrak{p}$, we deduce that $\mathfrak{p} + \langle a \rangle, \mathfrak{p} + \langle b \rangle \notin \Sigma_{\mathfrak{a},S}$. Since $\mathfrak{p} + \langle a \rangle, \mathfrak{p} + \langle b \rangle$ are ideals of A that contain $\mathfrak{p} \supseteq \mathfrak{a}$ as

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a subset, we deduce that $\mathfrak{p}+\langle a\rangle \cap S\neq\emptyset$ and $\mathfrak{p}+\langle b\rangle\cap S\neq\emptyset$.

Say $s_1\in\mathfrak{p}+\langle a\rangle$ and $s_2\in\mathfrak{p}+\langle b\rangle$. Then

$$\left.\begin{array}{l}\exists p_1\in\mathfrak{p}, r_1\in A, \quad s_1=p_1+ar_1\\ \exists p_2\in\mathfrak{p}, r_2\in A, \quad s_2=p_2+br_2\end{array}\right\} \Rightarrow S\ni s_1s_2=\overbrace{\boxed{p_1}(p_2+br_2)}^{\text{in }\mathfrak{p}}+\underbrace{(ar_1)\boxed{p_2}}_{\text{in }\mathfrak{p}}+\underbrace{(ar_1)(br_2)}_{\substack{\text{in }\mathfrak{p}\\ \text{as } ab\in\mathfrak{p}}}$$

$\Rightarrow s_1s_2\in\mathfrak{p}\cap S$ which contradicts $\mathfrak{p}\in\Sigma_{\mathfrak{a},S}$. ■

<u>Def.</u>. $\operatorname{Nil}(A):=\{a\in A \mid a^n=0 \text{ for some } n\in\mathbb{Z}^+\}$

is called the nilradical of A; and $a\in A$ is called nilpotent if $a^n=0$ for some $n\in\mathbb{Z}^+$. (So $\operatorname{Nil}(A)$ consists of nilpotent elements of A.)

<u>Theorem</u>. (1) $\operatorname{Nil}(A)\trianglelefteq A$

(2) $\operatorname{Nil}(A)=\bigcap_{\mathfrak{p}\in\operatorname{Spec}A}\mathfrak{p}$.

Pf. (1) • $a,b\in\operatorname{Nil}(A)\Rightarrow\exists n,m\in\mathbb{Z}^+$, $a^n=b^m=0$

$$\Rightarrow (a+b)^{n+m}=\sum_{i=0}^{n+m}\binom{n+m}{i}a^i b^{n+m-i}=0$$

$$\boxed{\begin{array}{ccc}\text{either } i\geq n & \text{or} & n+m-i\geq m,\\ \Downarrow & & \Downarrow\\ a^i=0 & & b^{n+m-i}=0\end{array}}$$

$\Rightarrow a+b\in\operatorname{Nil}(A)$

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. $a \in \mathrm{Nil}(A)$, $r \in A \Rightarrow \left.\begin{cases} a^n = 0 & \text{for some } n \in \mathbb{Z}^+ \\ r \in A \end{cases}\right\} \Rightarrow (ra)^n = 0$

$\Rightarrow ra \in \mathrm{Nil}(A)$.

(2). $a \in \mathrm{Nil}(A) \Rightarrow \exists n \in \mathbb{Z}^+,\ a^n = 0$

$\Rightarrow \forall \mathfrak{p} \in \operatorname{Spec} A,\ a^n \in \mathfrak{p}$

$\Rightarrow$ By induction on n, $a \in \mathfrak{p}$

$\boxed{a^n = a \cdot a^{n-1} \in \mathfrak{p} \Rightarrow \text{either } a \in \mathfrak{p} \text{ or } a^{n-1} \in \mathfrak{p}.}$

$\Rightarrow a \in \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}$.

. Suppose to the contrary that $\exists\ a \in \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} \setminus \mathrm{Nil}(A)$. (*)

Then $S_a := \{1, a, a^2, \ldots\}$ is a multiplicatively closed set that does not contain 0. So by the previous theorem $\exists \mathfrak{p}_0 \in \operatorname{Spec} A \cap \Sigma_{0, S_a}$; but this means $\mathfrak{p}_0 \cap S_a = \emptyset$, which implies $a \notin \mathfrak{p}_0$. And this contradicts (*). ∎