1 Homework 2.

1. Suppose G is a simple group and it has a subgroup H of index n where n is an integer more than 1. Prove that G can be embedded into the symmetric group S_n.

2. For a group G, let $\mathrm{Aut}(G)$ be the group of automorphisms of G. Let $c : G \to \mathrm{Aut}(G), c(g) := c_g$, where $c_g(x) := gxg^{-1}$ for every $x \in G$.

   (a) Prove that c_g is an automorphism of G and c is a group homomorphism.

   (b) Prove that $\ker c$ is the center $Z(G)$ of G; recall that

   $$Z(G) := \{g \in G \mid \forall x \in G, gx = xg\}.$$

   (c) The image of c is called the group of *inner automorphisms* of G, and it is denoted by $\mathrm{Inn}(G)$. Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

   (d) Prove that $|Z(\mathrm{Aut}(G))| \le |\mathrm{Hom}(G, Z(G))|$; in particular, if either $Z(G) = 1$ or G is perfect (that means G is equal to its derived subgroup $[G, G]$), then $Z(\mathrm{Aut}(G)) = \{1\}$.

   (**Hint.** The following statements can be useful.

   (a) For all $g \in G$ and $\phi \in \mathrm{Aut}(G)$, $\phi \circ c_g \circ \phi^{-1} = c_{\phi(g)}$.

   (b) For $\phi \in Z(\mathrm{Aut}(G))$, $c_g = c_{\phi(g)}$; and so $\phi(g) = g\eta(g)$ for some $\eta(g)$ in $Z(G)$.

   (c) If $\eta : G \to Z(G)$ is the function given in the previous statement, then η is a group homomorphism.)

3. Let $\mathrm{SL}_2(\mathbb{R})$ be the set of real 2-by-2 matrices with determinant 1. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathscr{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, let

   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}.$$

(a) Prove that $\mathrm{Im}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z\right) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$.

(b) Prove that $\cdot$ is an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathscr{H}$.

4. Suppose G is a finite group, $C \subseteq \mathbb{R}^n$ is a convex subset; that means, for all two points P, Q in C, the segment PQ is a subset of C. Suppose G acts on C by affine transformations; that means

$$\forall P, Q \in C, \forall t \in [0,1], \forall g \in G, \quad g \cdot (tP + (1-t)Q) = tg \cdot P + (1-t)g \cdot Q.$$

Prove that G has a fixed point; that means there exists $x \in C$ such that, for all $g \in G$, $g \cdot x = x$.

(**Hint.** 1. Using induction and convexity of C, prove that for every $c_1, \ldots, c_n \in C$, their average is in C:

$$\frac{c_1 + \ldots + c_n}{n} \in C.$$

2. For $y \in C$, consider the average $A_G(y)$ of the points in $\{g \cdot y \mid g \in G\}$. Prove that $A_G(y)$ is a fixed point of G.)

5. Suppose G is a finite subgroup of the group $\mathrm{GL}_n(\mathbb{R})$ of n-by-n invertible real matrices. Prove that there is a G-*invariant inner product* on $\mathbb{R}^n$.

(**Hint.** An *inner product* is a function $\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with the following properties:

(a) (Bilinear) For all $c_1, c_2 \in \mathbb{R}$ and $v, v_1, v_2, w, w_1, w_2 \in \mathbb{R}^n$,

$$\langle c_1 v_1 + c_2 v_2, w\rangle = c_1\langle v_1, w\rangle + c_2\langle v_2, w\rangle, \;\; \langle v, c_1 w_1 + c_2 w_2\rangle = c_1\langle v, w_1\rangle + c_2\langle v, w_2\rangle.$$

(b) (Symmetric) For all $v, w \in \mathbb{R}^n$, $\langle v, w\rangle = \langle w, v\rangle$.

(c) (Positive definite) For all $v \in \mathbb{R}^n$, $\langle v, v\rangle > 0$.

For instance,

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) := \sum_{i=1}^{n} a_i b_i$$

is an inner product on $\mathbb{R}^n$.

Use *taking the average* technique: for $v, w \in \mathbb{R}^n$, let

$$\langle v, w\rangle := \frac{1}{|G|}\sum_{g\in G}(gv)\cdot(gw).$$

Prove that $\langle , \rangle$ is a G-invariant inner product.)

(**Remark**. This problem plays an important role in representation theory of finite groups. Here is an application of this exercise. Suppose $V \subseteq \mathbb{R}^n$ is a subspace which is invariant under G; that means for all $v \in V$ and $g \in G$, $gv \in V$. Suppose $\langle , \rangle$ is a G-invariant inner product. Then

$$V^\perp := \{w \in \mathbb{R}^n \mid \forall v \in V, \langle w, v\rangle = 0\}$$

is also a G-invariant subspace and $\mathbb{R}^n = V \oplus V^\perp$.)

6. Suppose H is a subgroup of G. Let

$$C_G(H) := \{x \in G \mid \forall h \in H\}$$

be the *centralizer* of H in G, and

$$N_G(H) := \{x \in G \mid xHx^{-1} = H\}$$

be the *normalizer* of H in G. Both of these are known to be subgroups of G and clearly $C_G(H) \subseteq N_G(H)$. Prove that $N_G(H)/C_G(H)$ can be embedded into $\mathrm{Aut}(H)$.

(**Hint.** Notice that $N_G(H)$ acts on H by conjugation, which gives us a group homomorphism from $N_G(H)$ to $\mathrm{Aut}(H)$.)

7. Suppose N is a finite cyclic normal subgroup of G. Prove that every subgroup of N is normal in G.