1 Homework 3.

1. Suppose a finite group G acts on a finite set X.

   (a) Prove that $|{}_G\backslash^X| = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|$.

   (b) Suppose $|X| > 1$ and the action $G \curvearrowright X$ is transitive; that means there is only one orbit. Prove that there exists an element $g \in G$ with no fixed points.

   (c) Suppose H is a proper subgroup of G. Prove that G is not $\bigcup_{x\in G} xHx^{-1}$.

   (d) Are there an infinite group G and a proper subgroup H of G such that $G = \bigcup_{x\in G} xHx^{-1}$?

   Hint. (a) Consider the set

   $$A := \{(g,x) \in G \times X \mid g\cdot x = x\},$$

   and count the number of elements of this set in two ways. First fix x and count over g, and deduce that

   $$|A| = \sum_{x\in X}|G_x|.$$

   Next, fix g and count over x, and obtain that

   $$|A| = \sum_{g\in G}|\mathrm{Fix}(g)|.$$

   Now, notice that $|G_x| = |G'_x|$ if x and x' are in the same orbit $\mathscr{O}$. Hence $|G_x| = n(\mathscr{O}_x)$ only depends on the G-orbit of x. Therefore,

   $$\sum_{x\in X}|G_x| = \sum_{\mathscr{O}\in{}_G\backslash^X}\sum_{x\in\mathscr{O}} n(\mathscr{O}) = \sum_{\mathscr{O}\in{}_G\backslash^X}|\mathscr{O}|\,n(\mathscr{O}).$$

   (b) Use part (a). (c) Consider the transitive action $G \curvearrowright G/H$ by left-translations. (d) Use linear algebra to show that every element of $\mathrm{GL}_2(\mathbb{C})$ has a conjugate that is an upper triangular matrix.

2. Suppose $p < q < \ell$ are three primes, G is a group, and $|G| = pq\ell$. Then G has a normal Sylow ℓ-subgroup.

   (**Hint.** First prove that G has a normal subgroup of order either p, q, or ℓ elements.)

3. Suppose G is a finite group, N is a normal subgroup of G, and $P \in \mathrm{Syl}_p(N)$. Then $G = N_G(P)N$.

   (**Hint**. For every $g \in G$, argue that gPg^{-1} is a Sylow p-subgroup of N. Use the fact that every two Sylow p-subgroups of N are conjugate in N.)

4. Suppose G is a finite group and H is a subgroup. Suppose for all $x \in H \backslash \{1\}$, $C_G(x) \subseteq H$. Prove that $\gcd(|H|, [G:H]) = 1$.

   (**Hint.** Suppose p is a prime which divides $\gcd(|H|, [G:H])$. Suppose $Q \in \mathrm{Syl}_p(H)$. Argue that there exists $P \in \mathrm{Syl}_p(G)$ such that $Q \subseteq P$. Argue that there exists $y \in Z(Q) \setminus \{1\}$. Considering $C_G(y)$, show that $Z(P) \subseteq Q$. Suppose $x \in Z(P) \setminus \{1\}$, consider $C_G(x)$ to obtain that $P \subseteq H$. Argue why this is a contradiction.)

5. Suppose G is a finite group, N is a normal subgroup, and p is a prime factor of $|N|$.

   (a) Suppose $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_p(N)$. Prove that there exists $g \in G$ such that $Q = gPg^{-1} \cap N$.

   (b) Prove that the following is a well-defined surjective function

   $$\Phi : \mathrm{Syl}_p(G) \to \mathrm{Syl}_p(N), \quad \Phi(P) := P \cap N.$$

   (c) For $P \in \mathrm{Syl}_p(G)$, prove that $N_G(P) \subseteq N_G(\Phi(P))$ and

   $$|\Phi^{-1}(\Phi(P))| = [N_G(\Phi(P)) : N_G(P)].$$

   (d) Prove that $|\mathrm{Syl}_p(N)|$ divides $|\mathrm{Syl}_p(G)|$.

   (**Hint**. Notice that we have $\Phi(gPg^{-1}) = g\Phi(P)g^{-1}$ for every $g \in G$ and $P \in \mathrm{Syl}_p(G)$. Use this to obtain that $[N_G(\Phi(P)) : N_G(P)]$ does not depend on the choice of P.)

6. Suppose p is an odd prime and G is a group of order $p(p+1)$ which does not have a normal subgroup of order p. Prove that p is a Mersenne prime; that means $p = 2^n - 1$ for some positive integer n.

   (**Hint**. Go through the proof in the lecture note.)

7. Suppose p and q are prime numbers and G is a group of order p^2q. Prove that G is not simple.