1 Homework 7.

1. Suppose G is a group. For all $x, y \in G$, let

$$[x, y] := xyx^{-1}y^{-1} \quad \text{and} \quad {}^{x}y := xyx^{-1}.$$

Then Hall's equation asserts that

$$[[x, y], {}^{y}z][[y, z], {}^{z}x][[z, x], {}^{x}y] = 1$$

for all $x, y, z \in G$. You can check this on your own and use it in this exercise.

   (a) Suppose H, K, L are normal subgroups of G. Prove that

$$[[H, K], L] \leq [[K, L], H][[L, H], K].$$

   (b) Prove that for every positive integers m and n,

$$[\gamma_m(G), \gamma_n(G)] \subseteq \gamma_{m+n}(G).$$

(**Hint.** (1) Since H, K, L are normal subgroups,

$$[[K, L], H][[L, H], K]$$

is a normal subgroup of G. Consider $\overline{G} := G/[[K, L], H][[L, H], K]$, let $\overline{H}$, $\overline{K}$, and $\overline{L}$ be the quotient of H, L, and K by $[[K, L], H][[L, H], K]$. Use Hall's equation, and obtain that for all $h \in \overline{H}$, $k \in \overline{K}$, and $l \in \overline{L}$, we have that $[h, k]$ and l commute. Deduce that $[\overline{H}, \overline{K}]$ commute with l. Obtain that $[[\overline{H}, \overline{K}], \overline{L}] = 1$.

(2) Use induction on m and part (a).)

2. The Frattini subgroup $\Phi(G)$ of a group G is the intersection of all of its maximal subgroups. Suppose G is a finite group. Notice that under an automorphism of G, a maximal subgroup is sent to a maximal subgroup, and so the Frattini subgroup $\Phi(G)$ is a characteristic subgroup.

   (a) Suppose H is a subgroup of G and $H\Phi(G) = G$. Prove that $H = G$.

   (b) Suppose $S \subseteq G$. Prove that $\langle S \rangle = G$ if and only if $\langle \pi(S) \rangle = G/\Phi(G)$ where $\pi : G \to G/\Phi(G)$ is the natural quotient map.

(c) Prove that $\Phi(G)$ is nilpotent.

(d) Prove that G is nilpotent if and only if $G/\Phi(G)$ is nilpotent.

(**Hint.** (1) Suppose to the contrary that H is a proper subgroup, and let M be a maximal subgroup of G which contains H. Argue why $H\Phi(G) \subseteq M$.

(2) Suppose P is a Sylow p-subgroup of $\Phi(G)$. Argue why $G = N_G(P)\Phi(G)$ (The Frattini argument). Deduce that $G = N_G(P)$.

(3) Suppose Q is a Sylow p-subgroup of G. Assuming $G/\Phi(G)$ is nilpotent, deduce that $Q\Phi(G)$ is a normal subgroup of G. Use the Frattini argument and show that $N_G(Q)Q\Phi(G) = G$. Obtain that $N_G(Q) = G$.)

3. Suppose P is a finite group and $|P| = p^n$ where p is prime and n is a positive integer. Let $\mathrm{Max}(P)$ be the set of all maximal subgroups of P.

(a) Prove that for all $M \in \mathrm{Max}(P)$, $P/M \simeq \mathbb{Z}/p\mathbb{Z}$.

(b) Prove that $P/\Phi(P)$ is isomorphic to a subgroup of $(\mathbb{Z}/p\mathbb{Z})^{|\mathrm{Max}(P)|}$.

(c) Prove that $\Phi(P) = P^p[P,P]$ where

$$P^p[P,P] := \{x^p y \mid x \in P, y \in [P,P]\}.$$

(d) Suppose $P = \langle S \rangle$ and a proper subset of S does not generate P. Prove that $|S| = \dim_{\mathbb{Z}/p\mathbb{Z}}(P/P^p[P,P])$.

(**Hint**. (1) Since P is nilpotent, M is a proper subgroup of $N_G(M)$. Deduce that M is a normal subgroup. Use the correspondence theorem and obtain that G/M has no non-trivial subgroup. Deduce that $G/M \simeq \mathbb{Z}/p\mathbb{Z}$.

(2) Consider the group homomorphism,

$$\pi : P \to \prod_{M \in \mathrm{Max}(P)} P/M, \quad \pi(x) := (xM)_{M \in \mathrm{Max}(P)}$$

Argue why the kernel of π is $\Phi(P)$. Deduce that $P/\Phi(P)$ is isomorphic to a subgroup of $(\mathbb{Z}/p\mathbb{Z})^{|\mathrm{Max}(P)|}$.

(3) Use part (b) and deduce that $x^p \in \Phi(P)$ and $[x_1, x_2] \in \Phi(P)$ for all $x, x_1, x_2 \in P$. Obtain that $P^p[P,P] \subseteq \Phi(P)$.

(4) Notice that $P/[P,P]$ is an abelian group, and so $\overline{x} \mapsto \overline{x}^p$ is a group homomorphism from $P/[P,P]$ to itself. The image of this group homomorphism is $P^p[P,P]/[P,P]$. Hence $P^p[P,P]$ is a normal subgroup of P. Consider

$$V := P/P^p[P,P].$$

Notice that V is a finite abelian group and every non-trivial element has order p. Use the additive notation for V. Notice that V can be viewed as a vector space over $\mathbb{Z}/p\mathbb{Z}$. Then every non-zero element $v \in V$ is part of a $\mathbb{Z}/p\mathbb{Z}$-basis of V. Hence there is a subspace W of codimension 1 which does not contain v. Notice that $V/W \simeq \mathbb{Z}/p\mathbb{Z}$, and so W is a maximal subgroup of V. Deduce that for all $x \notin P^p[P,P]$ there is $M \in \mathrm{Max}(P)$ such that $x \notin M$. Hence $\Phi(P) \subseteq P^p[P,P]$.

(5) Notice that $P = \langle S \rangle$ implies that $\pi(S)$ generates $P/\Phi(P)$. By part (c), $P/\Phi(P)$ is a vector space over $\mathbb{Z}/p\mathbb{Z}$. Hence the $\mathbb{Z}/p\mathbb{Z}$-span of $\pi(S)$ is $P/\Phi(P)$. Therefore there is a $\mathbb{Z}/p\mathbb{Z}$-basis $\overline{S}'$ of $P/\Phi(P)$ which is a subset of $\pi(S)$. Suppose $S' \subseteq S$ is such that $|S'| = |\overline{S}'|$ and $\overline{S}' = \pi(S')$. Argue why $\langle S' \rangle = P$. Deduce that $S' = S$, and obtain that $|S| = \dim_{\mathbb{Z}/p\mathbb{Z}}(P/\Phi(P))$.)

(**Remark.** Part (d) implies that every two minimal generating set of P have the same cardinality. This statement is false if P is not a finite p-group. For instance $G := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is a cyclic group and so it can be generated by 1 element, but the set $\{(1,0),(0,1)\}$, which has 2 elements, is also a minimal generating set of G.)