1 Homework 8.

1. Suppose G is a finite group and H is a non-trivial subgroup of G.

   (a) Show that there exists a function $f : \mathrm{Syl}_p(H) \to \mathrm{Syl}_p(G)$ such that for all $\overline{P} \in \mathrm{Syl}_p(H)$, $\overline{P} = f(\overline{P}) \cap H$. Deduce that

   $$|\mathrm{Syl}_p(H)| \leq |\mathrm{Syl}_p(G)|.$$

   (b) Suppose G does not have a non-trivial normal p-subgroup. Suppose $\overline{P}$ is a non-trivial p-subgroup of G. Prove that

   $$|\mathrm{Syl}_p(N_G(\overline{P}))| < |\mathrm{Syl}_p(G)|.$$

   (**Hint.** To prove the second part, use the contrary assumption and deduce that f is surjective. Use the surjectivity of f and show that the intersection of all Sylow p-subgroups of G is a non-trivial subgroup.)

   (**Remark.** The intersection of all the Sylow p-subgroups of G is denoted by $O_p(G)$. This is the largest normal p-subgroup of G; this means $O_p(G) \trianglelefteq G$ and if $\overline{P}$ is a normal p-subgroup of G, then $\overline{P} \subseteq O_p(G)$. In the above argument, you are showing that if $|\mathrm{Syl}_p(H)| = |\mathrm{Syl}_p(G)|$, then

   $$O_p(H) = O_p(G) \cap H.)$$

2. Suppose G is a group of order $p^n q$ where p and q are primes. Prove that G is solvable.

   (**Hint.** Use the following steps.

   (a) First show that it is enough to argue why a group of order $p^n q$ is not a non-abelian simple group.

   (b) Suppose to the contrary that there exists a non-abelian simple group G of order $p^n q$.

   (c) Prove that G has exactly q Sylow p-subgroups.

   (d) Let $\overline{P}$ be maximal among the intersections of pairs of Sylow p-subgroups. Suppose $\overline{P} \neq \{1\}$, and let $H := N_G(\overline{P})$. Prove that H has at least 2 Sylow p-subgroups.

(e) Prove that $|\mathrm{Syl}_p(H)| = q$. Get a contradiction using Problem 1.

(f) Let $Q := G \setminus (\bigcup_{P\in \mathrm{Syl}_p(G)}(P \setminus \{1\})$. Prove that $|Q| = q$, and deduce that G has a unique Sylow q-subgroup. Get a contradiction.)

(**Remark.** Burnside used character theory of finite groups to prove that a group of order $p^n q^m$ is solvable.)

3. Suppose $G = \langle g_1, \ldots, g_n\rangle$ and H is a subgroup of index at most k.

   (a) Prove that H has a subgroup N such that $[G : N] \le k!$ and $N \trianglelefteq G$.

   (b) Suppose S is a finite subset of G such that

   $$\{g_1, \ldots, g_n\} \subseteq S$$

   and for all $g \in G$ there exists $s \in S$ such that

   $$gN = sN,$$

   and for all $s \in S$

   $$s^{-1} \in S.$$

   For all $s, s' \in S$, choose $c(s, s') \in S$ such that $c(s, s')N = ss'N$, and let $f(s, s') := c(s, s')^{-1}ss'$. Notice that the image of f is a finite subset S_N of N. Prove that $N = \langle S_N \cup (S \cap N)\rangle$.

   (c) Prove that a finite index subgroup of a finitely generated group is finitely generated.

(**Hint.** Let $\overline{N}$ be the group generated by the image of f. Prove that for all $s_1, \ldots, s_r \in S$,

$$s_{i_1} \cdots s_{i_k}\overline{N} = s_{i_1} \cdots s_{i_{k-2}} s'\overline{N}$$

for some $s' \in S$. Take $n \in N$, use the previous argument to deduce that $n\overline{N} = s\overline{N}$ for some $s \in S$.)

4. Suppose G is a group. Let $t_n(G)$ be the number of transitive actions of G on $[1..n]$ and

   $$a_n(G) := \{H \le G \mid [G : H] = n\}.$$

   (a) Prove that $t_n(G) = a_n(G)(n-1)!$.

(b) Suppose G is a finitely generated subgroup. Prove that for all positive integers n, G has only finitely many subgroups of index at most n.

(**Hint.** For the second part, think about $\mathrm{Hom}(G, S_n)$.)

(**Remark.** Lubotzky initiated the study of *subgroup growth* of a finitely generated group.)

5. Suppose $G = \langle a, b \rangle$ and $a^2 = b^2 = 1$.

   (a) Prove that G is solvable.

   (b) Show that G is not necessarily finite.

   (**Hint.** Let $N = \langle ab \rangle$. Prove that $a, b \in N_G(N)$ and deduce that $N \trianglelefteq G$.)

6. Suppose G is a group and $a, b \in G$. Suppose

$$ab^2a^{-1} = b^3 \quad \text{and} \quad ba^2b^{-1} = a^3.$$

   Prove that $a = b = 1$.

   (**Hint.** Consider $a^2b^4a^{-2}$ and prove that $a \in C_G(b^4)$.)

   (**Remark.** Given a set of relations R and two words w_1 and w_2, we can ask whether there exists an algorithm to decide if $w_1 = w_2$. This is called *the word problem*. Novikov proved that in general the answer to this question is negative. This is in contrast to vector spaces, where we can use the reduced row process and find out if a vector is 0. The concept of Gröbner basis gives us an affirmative answer to the commutative ring theoretic analogue of this question.)

7. Suppose A is a unital ring (not necessarily commutative). Suppose $\mathfrak{a}$ is an ideal of A and $\mathfrak{a}^n = 0$ for some positive integer n; that means

$$x_1 \cdots x_n = 0$$

   for all $x_1, \ldots, x_n \in \mathfrak{a}$.

   (a) Let $N := 1 + \mathfrak{a}$. Prove that N is a nilpotent group under multiplication.

   (b) Suppose R is a unital commutative ring, and $U_n(R)$ is the set of all n-by-n upper-triangular matrices with entries in R and diagonal entries equal to 1. Prove that $U_n(R)$ is a nilpotent group.

(c) Prove that the set $B_n(R)$ of all n-by-n upper-triangular matrices with entries in R and diagonal entries in the group of units $R^\times$ of R is a solvable group.

(**Hint.** For the first part, notice that for all $a \in \mathfrak{a}$,

$$(1-a)(1+a+\cdots+a^{n-1}) = 1,$$

and so every element of $1+\mathfrak{a}$ is invertible. Inductively argue why

$$\gamma_i(N) \subseteq 1+\mathfrak{a}^i,$$

where

$$\mathfrak{a}^i := \langle a_1 \cdots a_i \mid a_1, \ldots, a_i \in \mathfrak{a} \rangle$$

(the smallest ideal which contains all these elements). For the second part, let A be the set of all n-by-n upper-triangular matrices with entries in R. You are allowed to use without proof that A is a unital ring. Let $\mathfrak{a}$ be the set of all the elements of A that have zero diagonal entries. Argue why $\mathfrak{a}$ is an ideal of A and $\mathfrak{a}^n = 0$. For the third part, argue why

$$[B_n(R), B_n(R)] \subseteq U_n(R).)$$