1 Homework 3.

1. Suppose $p < q < \ell$ are three primes, G is a group, and $|G| = pq\ell$. Then G has a normal Sylow ℓ-subgroup.

   (**Hint.** First prove that G has a normal subgroup of order either p, q, or ℓ elements.)

2. Suppose G is a finite group, N is a normal subgroup of G, and $P \in \mathrm{Syl}_p(N)$. Then $G = N_G(P)N$.

   (**Hint**. For every $g \in G$, argue that gPg^{-1} is a Sylow p-subgroup of N. Use the fact that every two Sylow p-subgroups of N are conjugate in N.)

3. Suppose G is a finite group and H is a subgroup. Suppose for all $x \in H\backslash\{1\}$, $C_G(x) \subseteq H$. Prove that $\gcd(|H|, [G : H]) = 1$.

   (**Hint.** Suppose p is a prime which divides $\gcd(|H|, [G : H])$. Suppose $Q \in \mathrm{Syl}_p(H)$. Argue that there exists $P \in \mathrm{Syl}_p(G)$ such that $Q \subseteq P$. Argue that there exists $y \in Z(Q) \setminus \{1\}$. Considering $C_G(y)$, show that $Z(P) \subseteq Q$. Suppose $x \in Z(P) \setminus \{1\}$, consider $C_G(x)$ to obtain that $P \subseteq H$. Argue why this is a contradiction.)

4. Suppose G is a finite group, N is a normal subgroup, and p is a prime factor of $|N|$.

   (a) Suppose $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_p(N)$. Prove that there exists $g \in G$ such that $Q = gPg^{-1} \cap N$.

   (b) Prove that the following is a well-defined surjective function

   $$\Phi : \mathrm{Syl}_p(G) \to \mathrm{Syl}_p(N), \quad \Phi(P) := P \cap N.$$

   (c) For $P \in \mathrm{Syl}_p(G)$, prove that $N_G(P) \subseteq N_G(\Phi(P))$ and

   $$|\Phi^{-1}(\Phi(P))| = [N_G(\Phi(P)) : N_G(P)].$$

   (d) Prove that $|\mathrm{Syl}_p(N)|$ divides $|\mathrm{Syl}_p(G)|$.

   (**Hint**. Notice that we have $\Phi(gPg^{-1}) = g\Phi(P)g^{-1}$ for every $g \in G$ and $P \in \mathrm{Syl}_p(G)$. Use this to obtain that $[N_G(\Phi(P)) : N_G(P)]$ does not depend on the choice of P.)

5. Suppose p is an odd prime and G is a group of order $p(p+1)$ which does not have a normal subgroup of order p. Prove that p is a Mersenne prime; that means $p = 2^n - 1$ for some positive integer n.

   (**Hint**. Go through the proof in the lecture note.)

6. Suppose p and q are prime numbers and G is a group of order p^2q. Prove that G is not simple.

7. A subgroup K of G is called a *characteristic* subgroup if for all $\theta \in \mathrm{Aut}(G)$, $\theta(K) = K$. Notice that every characteristic subgroup is normal.

   (a) Suppose N is a normal subgroup of G and K is a characteristic subgroup of N. Prove that K is a normal subgroup of G.

   (b) We say a group H is *characteristically simple* if the only characteristic subgroups of H are 1 and H. Suppose N is a *minimal normal* subgroup of G; that means if $M \leq N$ and $M \trianglelefteq G$, then either $M = \{1\}$ or $M = N$. Then N is characteristically simple.

8. Suppose G is a finite group.

   (a) Prove that a normal Sylow p-subgroup is a characteristic subgroup.

   (b) Suppose H is a normal subgroup of G and $\gcd(|H|, [G : H]) = 1$. Prove that H is a characteristic subgroup.

   (**Hint**.These parts are not related to each other. For the second part, suppose $\theta(H) \neq H$ for some $\theta \in \mathrm{Aut}(G)$. show that $|\theta(H)H/H|$ divides $|\theta(H)|$ and $|G/H|$.)