1 Homework 4.

1. Suppose D_{2n} is a dihedral group. Prove that there exists a splitting SES of the form $1 \to C_n \to D_{2n} \to C_2 \to 1$ where C_k is the cyclic group of order k.

2. Suppose G is a group.

   (a) Show that if N_1 and N_2 are normal subgroups of G and $N_1 \cap N_2 = \{1\}$, then for all $x_1 \in N_1$ and $x_2 \in N_2$, $x_1 x_2 = x_2 x_1$.

   (b) Suppose $N_1, \ldots, N_k$ are normal subgroups of G and $N_i \cap N_i = \{1\}$ for all $i \neq j$. Prove that

   $$f : \prod_{i=1}^{k} N_i \to N_1 \cdots N_k, \quad f(x_1, \ldots, x_k) := x_1 \cdots x_k$$

   is a group homomorphism.

   (c) Suppose $N_1, \ldots, N_k$ are normal subgroups of G, and for all i,

   $$N_i \cap N_1 \cdots N_{i-1} N_{i+1} \cdots N_k = \{1\}.$$

   Prove that

   $$f : \prod_{i=1}^{k} N_i \to N_1 \cdots N_k, \quad f(x_1, \ldots, x_k) := x_1 \cdots x_k$$

   is a group isomorphism.

3. Suppose in a finite group G for every proper subgroup H, $H \subsetneq N_G(H)$.

   (a) Prove that all the Sylow subgroups of G are normal. Deduce that for all prime divisors of $|G|$, G has a unique Sylow p-subgroup.

   (b) Prove that $G \simeq \prod_{p \text{ prime factor of } |G|} P_p$ where P_p is the unique Sylow p-subgroup of G.

   (**Hint.** Use $N_G(N_G(P)) = N_G(P)$ for every Sylow subgroup P, and the previous problem.)

4. Suppose G is a finite group and A is a normal abelian subgroup of G. Let $s : G/A \to G$ be a *section* of the natural projection map; that means for all $h \in G/A$, we choose an element $s(h)$ from the coset h. Alternatively we can say that $s(h)A = h$. Notice that if s is a group homomorphism, then the standard SES $1 \to A \to G \to G/A \to 1$ splits. The goal of this exercise is to modify s and make it into a group homomorphism under suitable assumptions. Let $H := G/A$ and define the function

$$c : H \times H \to A, \quad c(h_1, h_2) := s(h_1)s(h_2)s(h_1h_2)^{-1}.$$

Notice that since $s(h_1h_2)A = h_1h_2 = s(h_1)As(h_2)A = s(h_1)s(h_2)A$, the image of c is indeed in A. Function c gives us an insight on how far s is from being a group homomorphism. Notice that since A is abelian, the conjugation action of G on A factors through an action of H. More precisely, for all $h \in H$ and $a \in A$, let

$$h \cdot a := s(h)as(h)^{-1},$$

and notice that this is a well-defined group action.

(a) Prove that, for all $h_1, h_2, h_3 \in H$, we have

$$c(h_1, h_2)c(h_1h_2, h_3) = (h_1 \cdot c(h_2, h_3))c(h_1, h_2h_3).$$

(Since A is abelian, it is more customary to write this equation in an additive notation:

$$c(h_1, h_2) + c(h_1h_2, h_3) = h_1 \cdot c(h_2, h_3) + c(h_1, h_2h_3),$$

and this is called the 2-cocycle relation.)

(b) Prove that the standard SES $1 \to A \to G \to H \to 1$ splits if and only if there exists a function $b : H \to A$ such that

$$c(h_1, h_2) = b(h_1)(h_1 \cdot b(h_2))b(h_1h_2)^{-1}.$$

(Again it is customary to write this equation in an additive notation:

$$c(h_1, h_2) = b(h_1) + h_1 \cdot b(h_2) - b(h_1h_2).$$

This is called a 2-coboundary.)

(c) In the above setting, assume that $\gcd(|A|, |H|) = 1$. Prove that every 2-cycle is a 2-boundary. Deduce that the standard SES

$$1 \to A \to G \to H \to 1$$

splits.

(**Hint.**

(a) Since $s(h_1)s(h_2) = c(h_1, h_2)s(h_1h_2)$, we have

$$\begin{aligned}(s(h_1)s(h_2))s(h_3) =& c(h_1, h_2)s(h_1h_2)s(h_3)\\ =& c(h_1, h_2)c(h_1h_2, h_3)s(h_1h_2h_3).\end{aligned}$$

We also have

$$\begin{aligned}s(h_1)(s(h_2)s(h_3)) =& s(h_1)c(h_2, h_3)s(h_2h_3) = (h_1 \cdot c(h_2, h_3))s(h_1)s(h_2h_3)\\ =& (h_1 \cdot c(h_2, h_3))c(h_1, h_2h_3)s(h_1h_2h_3).\end{aligned}$$

(b) Notice that this SES splits if and only if there exists a function $b : H \to A$ such that $\psi(h) := b(h)^{-1}s(h)$ is a group homomorphism. For all $h_1, h_2 \in H$, we have

$$\begin{aligned}\psi(h_1)\psi(h_2)\psi(h_1h_2)^{-1} =& b(h_1)^{-1}s(h_1)b(h_2)^{-1}s(h_2)s(h_1h_2)^{-1}b(h_1h_2)\\ =& b(h_1)^{-1}(h_1 \cdot b(h_2))^{-1}c(h_1, h_2)b(h_1h_2).\end{aligned}$$

Hence, the given SES splits precisely when

$$c(h_1, h_2) = (h_1 \cdot b(h_2))b(h_1)b(h_1h_2)^{-1}.$$

(c) Use the additive notation for the abelian group A. Since $\gcd(|A|, |H|) = 1$, for every $a \in A$ there exists a unique $y \in A$ such that $|H|y = a$. Denote this element by $\frac{a}{|H|}$. Suppose c is a 2-cocyle and let

$$b : H \to A, \quad b(x) := \frac{\sum_{h \in H} c(x, h)}{|H|}.$$

Adding over the h_3 term in the 2-cocycle relation, deduce that

$$|H|c(h_1, h_2) + |H|b(h_1h_2) = |H|(h_1 \cdot b(h_2)) + |H|b(h_1),$$

and so c is a 2-coboundary.)

5. In this problem, you will show that S_6 has an automorphism which is not an inner automorphism.

   (a) Show that S_5 has 6 Sylow 5-subgroups.

   (b) Use the action of S_5 on $\mathrm{Syl}_5(S_5)$ and show that S_6 has a subgroup H which is isomorphic to S_5 and for every $\sigma \in S_6$, $\mathrm{Fix}(\sigma H \sigma^{-1}) = \varnothing$ where S_6 acts on $\{1, \ldots, 6\}$.

   (c) Consider the action $S_6 \curvearrowright S_6/H$ by left-translations. Argue that this action induces a group homomorphism $\theta : S_6 \to S_6$. Prove that $\mathrm{Fix}(\theta(H)) \neq \varnothing$.

   (d) Deduce that $\mathrm{Aut}(S_6) \neq \mathrm{Inn}(S_6)$.

   (In this problem you are allowed to use the fact that if N is a normal subgroup of S_n, $[S_n : N] > 2$, and $n \geq 5$, then $N = \{1\}$.)

6. Prove that a group of order 36 is not simple.

   (**Hint.** Suppose G is simple. Find the number of Sylow 3-subgroups of G. Consider the action of G on $\mathrm{Syl}_3(G)$. Prove that the kernel of this action cannot be trivial.)

7. Suppose N and H are two groups and $f_1, f_2 : H \to \mathrm{Aut}(N)$ are two group homomorphisms.

   (a) Suppose $\theta : N \rtimes_{f_1} H \to N \rtimes_{f_2} H$ is an isomorphism such that the following is a commutative diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \xrightarrow{\alpha_1} & N \rtimes_{f_1} H & \longrightarrow & H & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{id}_N} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \mathrm{id}_H} & & \\
1 & \longrightarrow & N & \xrightarrow{\beta_1} & N \rtimes_{f_2} H & \longrightarrow & H & \longrightarrow & 1
\end{array}$$

   Let $\sigma : H \to \mathrm{Aut}(N)$, $\quad \sigma(h) := f_2(h) \circ f_1(h)^{-1}$. Prove that $\sigma(h)$ is an inner automorphism of N for all $h \in H$.

   (b) In the setting of part (a), prove that

$$\sigma(h_1 h_2) = \sigma(h_1) \circ f_1(h_1) \circ \sigma(h_2) \circ f_1(h_1)^{-1}.$$

(c) Suppose there exists $\overline{\theta} \in \mathrm{Aut}(H)$ such that $f_1 = f_2 \circ \overline{\theta}$. Prove that there exists an isomorphism $\theta : N \rtimes_{f_1} H \simeq N \rtimes_{f_2} H$ such that the following is a commutative diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \xrightarrow{\alpha_1} & N \rtimes_{f_1} H & \longrightarrow & H & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{id}_N} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \overline{\theta}} & & \\
1 & \longrightarrow & N & \xrightarrow{\beta_1} & N \rtimes_{f_2} H & \longrightarrow & H & \longrightarrow & 1
\end{array}$$

(**Hint.** For parts (a) and (b), argue that there exists a function $n : H \to N$ such that $\theta(1, h) = (n(h), h)$. Consider $\theta((1, h)(n, 1)(1, h)^{-1})$. For part (c), let

$$\theta : N \rtimes_{f_1} H \simeq N \rtimes_{f_2} H, \quad \theta(n, h) := (n, \overline{\theta}(h)).)$$