Name: ______________________________

PID: ______________________________

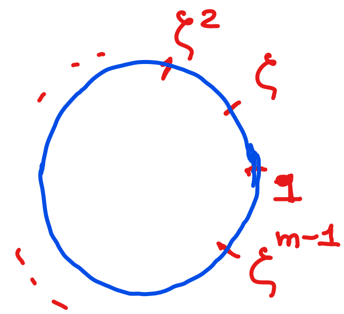
Question	Points	Score
1	10	
2	5	
3	5	
4	10	
5	10	
Total:	40	

1. Write your Name and PID, on the front page of your exam.
2. Read each question carefully, and answer each question completely.
3. Write your solutions clearly in the exam sheet.
4. Show all of your work; no credit will be given for unsupported answers.
5. You may use the result of one part of the problem in the proof of a later part, even you do not complete the earlier part.
6. You may use major theorems *proved* in class, but not if the whole point of the problem is reproduce the proof of a theorem proved in class or the textbook. Similarly, quote the result of a homework exercise only if the result of the exercise is a fundamental fact and reproducing the result of the exercise is not the main point of the problem.

1. (10 points) Prove that any element of finite order in $\mathrm{GL}_n(\mathbb{C})$ is diagonalizable.

Suppose $g \in GL_n(\mathbb{C})$ and $g^m = I$ for some $m \in \mathbb{Z}^+$. Then the minimal polynomial $m_g(t)$ of g divides $t^m - 1$. On the other hand, $t^m - 1 = (t-1)(t-\zeta)\cdots(t-\zeta^{m-1})$ where $\zeta = e^{\frac{2\pi i}{m}}$

And so $t^m - 1$ has distinct zeros; this implies $m_g(t)$ has distinct zeros. And so g is diagonalizable.



Let $\mathrm{diag}(J_{n_1}(\lambda_1), \ldots, J_{n_\ell}(\lambda_\ell))$ the the Jordan form of g. Then $J_{n_i}(\lambda_i)^m = I$. But $J_{n_i}(\lambda_i) = \lambda_i I + N_i$ and so

$$J_{n_i}(\lambda_i)^m = (\lambda_i I + N_i)^m = \sum_{j=0}^{m} \binom{m}{j} \lambda_i^{m-j} N_i^j = \sum_{j=0}^{n_i-1} \binom{m}{j} \lambda_i^{m-j} N_i^j$$

$$= \begin{bmatrix} \lambda_i^m & & & \\ m\lambda_i^{m-1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ \binom{m}{n_i-1}\lambda_i^{m-n_i+1} & \cdots & m\lambda_i^{m-1} & \lambda_i^m \end{bmatrix} \neq I \text{ unless } n_i = 1 \ (\text{and } \lambda_i^m = 1).$$

which means g is diagonalizable.

2. (5 points) Use the fact that $\mathbb{Z}[\sqrt{-10}]$ is not a UFD, and present a UFD D and a prime ideal $\mathfrak{p} \in \mathrm{Spec}(D)$ such that $D/\mathfrak{p}$ is not a UFD.

Let $\phi : \mathbb{Z}[x] \longrightarrow \mathbb{Z}[\sqrt{-10}]$ be the evaluation homomorphism $\phi(f(x)) := f(\sqrt{-10})$. Then ϕ is an onto homomorphism. And so by the 1^{st} isomorphism theorem $\mathbb{Z}[x]/_{\ker \phi} \simeq \mathbb{Z}[\sqrt{-10}]$; in particular this quotient is an integral domain; and so $\mathfrak{p} := \ker(\phi) \in \mathrm{Spec}(\mathbb{Z}[x])$.

Since $\mathbb{Z}$ is a UFD, $D := \mathbb{Z}[x]$ is a UFD.

3. Suppose A is a proper unital subring of $\mathbb{Z}[i]$ which is not $\mathbb{Z}$.

(a) (2 points) Prove that the field of fractions of A is $\mathbb{Q}[i]$. (Hint: think about it as a vector space over $\mathbb{Q}$)

Let F be the field of fractions of A. So $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}[i]$. Since A is unital subring, $\mathbb{Z} \subseteq A$. By our assumption $\mathbb{Z} \neq A$. And so $\exists\, a+ib \in A$ s.t. $b \neq 0$. Therefore $F \neq \mathbb{Q}$. Hence, as a vector space over $\mathbb{Q}$, $\dim_{\mathbb{Q}} F \geq 2$. Since $\mathbb{Q}[i] = \mathbb{Q} + \mathbb{Q}i$, $F \subseteq \mathbb{Q}[i]$, and $\dim_{\mathbb{Q}} F \geq 2$, we deduce that $F = \mathbb{Q}[i]$.

(b) (3 points) Prove that A is not a UFD.

From part (a) $\exists\, a_1, a_2 \in A$ s.t. $i = \frac{a_1}{a_2}$. And so $\left(\frac{a_1}{a_2}\right)^2 + 1 = 0$. If A is a UFD, then a_2 should divide the constant term of x^2+1; this means $a_2 \in A^\times$ and $i \in A$. But this means $\mathbb{Z}[i] \subseteq A$, which contradicts the assumption that $A \subsetneq \mathbb{Z}[i]$.

(A is NOT integrally closed, and so it cannot be a UFD.)

4. (10 points) Let D be a PID and M be a finitely generated D-module. For any $\mathfrak{p} \in \operatorname{Max}(D)$, let $k(\mathfrak{p}) := D/\mathfrak{p}$. Notice that $M/\mathfrak{p}M$ is a $k(\mathfrak{p})$-vector space (You do not need to prove this). Let $d(M)$ be the minimum number of generators of M (as a D-module). Prove that

$$d(M) = \max_{\mathfrak{p}\in\operatorname{Max}(D)} \dim_{k(\mathfrak{p})} M/\mathfrak{p}M.$$

By the classification of f.g. modules over a PID, we have $M \simeq D^r \oplus \bigoplus_{i=1}^{m} D/\langle a_i\rangle$ for some $a_1 | \cdots | a_m$ $(a_1 \notin D^*)$. In this case M can be generated by $\{e_j\}_{j=1}^{m+r}$ where $e_j = (0,\dots,0,\underbrace{1}_{j^{th}},0,\dots)$ and 1 is either the unity of D or the unity of $D/\langle a_i\rangle$.

Hence $d(M) \le r+m$. Since $M/\mathfrak{p}M$ is a quotient of M,

$\dim_{k(\mathfrak{p})} M/\mathfrak{p}M \le d(M) \le r+m$ for any $\mathfrak{p} \in \operatorname{Max}(D)$. (I)

Suppose p is an irreducible factor of a_1. Then $\mathfrak{p} := \langle p\rangle$ is a maximal ideal of D (as D is a PID). And

$$M/\mathfrak{p}M \simeq D^r/\mathfrak{p}D^r \oplus \bigoplus_{i=1}^{m} D/(\mathfrak{p}+\langle a_i\rangle) \simeq (D/\mathfrak{p})^r \oplus \bigoplus_{i=1}^{m} D/\mathfrak{p} = k(\mathfrak{p})^{r+m}$$

$$\boxed{\mathfrak{p}D^r = \mathfrak{p}^r \text{ and } \langle a_i\rangle \subseteq \mathfrak{p} \text{ as } p | a_1 | a_i}$$

And so $\dim_{k(\mathfrak{p})} M/\mathfrak{p}M = r+m$ (II). (I), (II) imply the claim.

5. (a) (4 points) Let k be a field and $X \in M_n(k)$ be a nilpotent n-by-n matrix with entries in k; that means $X^m = 0$ for some positive integer m. Prove that $X^n = 0$.

The minimal polynomial $m_X(t)$ divides t^m. Since irreducible factors of the Characteristic polynomial $f_X(t)$ and the minimal poly. are the same, $f_X(t)$ is a power of t. Since $\deg f_X(t) = n$, we deduce $f_X(t) = t^n$. By Cayley-Hamilton's theorem

$$0 = f_X(X) = X^n.$$

(b) (1 point) Suppose A is an integral domain, and $X \in M_n(A)$ is nilpotent. Prove that $X^n = 0$.

A can be embedded into its field of fractions k.

So $X \in M_n(k)$. Hence by part (a) $X^n = 0$.

(c) (5 points) Suppose A is a reduced ring; that means the nil-radical $\mathrm{Nil}(A) = 0$ of A is zero. Suppose $X \in M_n(A)$ is nilpotent. Prove that $X^n = 0$. (Hint: think about $A/\mathfrak{p}$ where $\mathfrak{p} \in \mathrm{Spec}(A)$.

For any $\mathfrak{p}$, $\Phi_{\mathfrak{p}} : M_n(A) \longrightarrow M_n(A/\mathfrak{p})$,

$\Phi_{\mathfrak{p}}([a_{ij}]) = [a_{ij} + \mathfrak{p}]$ is a ring homomorphism. So $\Phi_{\mathfrak{p}}(X)$ is nilpotent. By part (b), $\Phi_{\mathfrak{p}}(X)^n = 0$. And so $\Phi_{\mathfrak{p}}(X^n) = 0$; this means $X^n \in M_n(\mathfrak{p})$; for any $\mathfrak{p} \in \mathrm{Spec}(A)$. Hence

$$X^n \in \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} M_n(\mathfrak{p}) = M_n\Big(\bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p}\Big) = M_n(\mathrm{Nil}(A)) = \{0\}.$$

Good Luck!