Name: ______________________________

PID: ______________________________

Question	Points	Score
1	5	
2	15	
3	10	
4	10	
5	5	
6	10	
7	5	
8	20	
Total:	80	

1. Write your Name and PID, on the front page of your exam.
2. Read each question carefully, and answer each question completely.
3. Write your solutions clearly in the exam sheet.
4. Show all of your work; no credit will be given for unsupported answers.
5. You may use the result of one part of the problem in the proof of a later part, even you do not complete the earlier part.
6. You may use major theorems *proved* in class, but not if the whole point of the problem is reproduce the proof of a theorem proved in class or the textbook. Similarly, quote the result of a homework exercise only if the result of the exercise is a fundamental fact and reproducing the result of the exercise is not the main point of the problem.

1. (5 points) Let A be a unital commutative ring. Suppose P_1 and P_2 are projective A-modules. Prove that $P_1 \otimes_A P_2$ is a projective module.

2. (a) (5 points) Let $D := \mathbb{Z}[\sqrt{-5}]$ and $\mathfrak{a} := \langle 3, 1 + \sqrt{-5} \rangle$. Prove that $\mathfrak{a}$ is not a principal ideal.

(b) (10 points) Let $D := \mathbb{Z}[\sqrt{-5}]$ and $\mathfrak{a} := \langle 3, 1+\sqrt{-5}\rangle$. Prove that $\mathfrak{a}$ is a projective D-module.

3. (10 points) Suppose A is a unital commutative ring, and $A^n \simeq A^m$ as A-modules. Prove that $m = n$.

4. (10 points) Give an example of a unital commutative ring A and its subring B such that A is Noetherian and B is not Noetherian. Justify your answer.

5. (5 points) Suppose D is an integral domain and M is a flat D-module. Prove that M is torsion-free.

6. (10 points) Prove that $x^p - x + 1 \in \mathbb{F}_p[x]$ is irreducible.

7. (5 points) Suppose F is a field, $f(x) \in F[x]$ is irreducible, and E is a splitting field of $f(x)$ over F. Suppose there is $\alpha \in E$ such that $f(\alpha) = f(2\alpha) = 0$. Prove that the characteristic of F is positive.

8. (a) (10 points) Suppose $F \subseteq \mathbb{C}$ is a subfield and p is a prime number. Suppose $\zeta_p \in F$, where $\zeta_p := e^{2\pi i/p}$ is a p-th root of unity. Prove that for any $a \in F$, $[F[\sqrt[p]{a}] : F]$ is either 1 or p, where $\sqrt[p]{a}$ is a zero of $x^p - a$.

(b) (10 points) Suppose $K/\mathbb{Q}[\zeta_p]$ is a finite Galois extension, and $a \in K$. Prove that there is a finite Galois extension $L/\mathbb{Q}[\zeta_p]$ such that $\sqrt[p]{a} \in L$ and $[L : K]$ is a power of p. (Hint: think about $\prod_{\sigma \in \mathrm{Gal}(K/\mathbb{Q}[\zeta_p])}(x^p - \sigma(a))$.)

Good Luck!