# Homework 8

Monday, March 5, 2018 8:20 AM

1. Prove that $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{3}]$ are not isomorphic.

2. Prove that $\mathbb{F}_p(x,y)/\mathbb{F}_p(x^p,y^p)$ is not a simple extension.

3. Suppose $f(x) \in \mathbb{Q}[x]$ is irreducible, $\deg f = p$ is prime, and $f$ has $p-2$ real and 2 non-real zeros in $\mathbb{C}$. Let $K$ be a splitting field of $f(x)$ over $\mathbb{Q}$. Prove that $\mathrm{Aut}(K/\mathbb{Q}) \simeq S_p$.

(<u>Hint</u>. Since $K/\mathbb{Q}$ is normal, complex conjugation gives us an element of $\mathrm{Aut}(K/\mathbb{Q})$. Let $\alpha \in K$ be a zero of $f$. Then

$p = [\mathbb{Q}[\alpha]:\mathbb{Q}] \mid [K:\mathbb{Q}] = |\mathrm{Aut}(K/\mathbb{Q})|$. So $\mathrm{Aut}(K/\mathbb{Q})$ has an element of order $p$. Now think about the action of $\mathrm{Aut}(K/\mathbb{Q})$ on the set of zeros of $f(x)$.)

4. Let $E/F$ be an algebraic extension. Let

$$E_{sep} := \{\alpha \in E \mid m_{\alpha,F}(x) \text{ is separable}\}.$$

Prove that (1) $E_{sep}$ is a field and $E_{sep}/F$ is a separable extension.

(2) If $\mathrm{char}(F) = p > 0$, then $\forall \alpha \in E$, $\exists k \in \mathbb{Z}^+$, $m_{\alpha,E_{sep}}(x) = x^{p^k} - \alpha^{p^k}$ in particular $\alpha^{p^k} \in E_{sep}$.

# Homework 8

Friday, March 9, 2018 12:05 AM

(Hint. • Suppose $\alpha,\beta\in E_{sep}$. Let $L$ be a splitting field of $m_{\alpha,F}(x)m_{\beta,F}(x)$ over $F$. Argue that $L/F$ is separable. Deduce $\alpha\pm\beta, \alpha\beta^{\pm1}\in E_{sep}$.

• $m_{\alpha,F}(x) = g_\alpha(x^{p^k})$ where $g_\alpha(x)\in F[x]$ is irred. and separable.

Deduce that $g_\alpha(x) = m_{\alpha^{p^k},F}(x)$ and so $\alpha^{p^k}\in E_{sep}$.)

5. Suppose $E/F$ is a normal extension. Let $E_{sep}$ be as in Problem 4. Prove that $E_{sep}/F$ is a Galois extension.

6. Suppose $F\subseteq E\subseteq K$ is a tower of algebraic field extensions. Prove

$$K/F \text{ is separable} \iff K/E \text{ and } E/F \text{ are separable.}$$

(Hint. $(\Leftarrow)$ Consider $K_{sep}$ as in problem 4. Deduce $K_{sep}\supseteq E$; and so $\forall \alpha\in K$, $m_{\alpha,K_{sep}}(x) \mid m_{\alpha,E}(x)$. Now use part (2) of problem 4.)

7. Suppose $\sigma\in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $F:=Fix(\sigma)$. Suppose $E/F$ is a finite Galois extension (for some subfield $E$ of an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$). Prove that $Gal(E/F)$ is a finite cyclic group.

8. Let $E\subseteq\mathbb{C}$ be a splitting field of $x^p-2$ over $\mathbb{Q}$ where $p$ is an odd prime. Prove that $Gal(E/\mathbb{Q})\simeq \mathbb{Z}/p\mathbb{Z}\rtimes(\mathbb{Z}/p\mathbb{Z})^\times$