# Lecture 10: Rank of a free module

Monday, January 29, 2018 12:21 AM

In the previous lecture we were proving:

<u>Prop.</u> Suppose $R$ is a unital commutative ring. If $R^n \simeq R^m$ as $R$-mod, then $n=m$.

<u>Pf</u>. Let $\mathfrak{m}$ be a maximal ideal of $R$, and let

$\mathfrak{m}R^n := \{\sum m_i \cdot v_i \mid m_i \in \mathfrak{m}, v_i \in R^n\}$. Then $\mathfrak{m}R^n = \mathfrak{m}^n$ (why?).

Suppose $\phi : R^n \to R^m$ is an $R$-mod. isomorphism, then

$$\phi(\mathfrak{m}R^n) = \{\phi(\sum m_i \cdot v_i) \mid m_i \in \mathfrak{m}, v_i \in R^n\}$$

$$= \{\sum m_i \cdot \phi(v_i) \mid m_i \in \mathfrak{m}, v_i \in R^n\}$$

$$= \mathfrak{m}R^m.$$

Hence $\phi$ induces an $R$-mod isomorphism

$$\Phi : R^n/\mathfrak{m}R^n \longrightarrow R^m/\mathfrak{m}R^m.$$

Since $\mathfrak{m}(R^k/\mathfrak{m}R^k) = 0$, we can and will view $R^k/\mathfrak{m}R^k$ as an $R/\mathfrak{m}$-mod. Hence there is an $R/\mathfrak{m}$-vector space isomorp.

$(R/\mathfrak{m})^n \to (R/\mathfrak{m})^m$. Therefore they have equal dimensions.

And so $n=m$. ∎

# Lecture 10: Rank and modules over a PID

Monday, January 29, 2018 12:31 AM

Def. (a) $M$: $A$-mod. We say $x_1, \dots, x_k \in M$ are $A$-linearly indep. if

$$\sum Ax_i \text{ is an internal direct sum.}$$

(b) For an $A$-mod $M$, we let

$$\operatorname{rank} M := \max.\left\{ k \in \mathbb{Z}^{\geq 0} \;\middle|\; \begin{array}{l}\exists x_1, \dots, x_k \in M \text{ that are} \\ A\text{-linearly independent}\end{array} \right\}.$$

Proposition. $A$: integral domain. Then $\operatorname{rank}(A^n) = n$.

Pf. Let $F$ be the field of fractions of $A$; and view $A^n$ as a subset of $F^n$.

Suppose $x_1, \dots, x_{n+1} \in A^n \subseteq F^n$. Then $\exists\, \alpha_1, \dots, \alpha_{n+1} \in F$ s.t.

$$\alpha_1 x_1 + \dots + \alpha_{n+1} x_{n+1} = 0 \quad \text{and} \quad \exists i,\ \alpha_i \neq 0.$$

By clearing the denominators, $\exists\, a_i \in A$ s.t.

$\sum a_i x_i = 0$ and $\exists\, a_j \neq 0$. Hence $x_1, \dots, x_{n+1}$ are not $A$-linearly independent. ∎

Remark. In your HW assignment you will see that, for a unital commutative ring $A$, $\operatorname{rank}(A^n) = n$.

# Lecture 10: modules over PIDs

Thursday, February 1, 2018 3:04 PM

<u>Theorem.</u> Let $D$ be a PID, and $M$ be a submodule of $D^n$. Then

(a) $M$ is a free $D$-mod.

(b) $\exists\, x_1,\ldots,x_n \in D^n$ and $a_1,\ldots,a_m \in D\setminus\{0\}$ such that

(b-1) $a_1 \mid a_2 \mid \cdots \mid a_m$.

(b-2) $D^n = \bigoplus_{i=1}^{n} D x_i$.

(b-3) $M = \bigoplus_{i=1}^{m} D a_i x_i$.

<u>A few comments:</u> • the above statement for $\underline{n=1}$ is equivalent to saying $D$ is a PID.

• <u>Rough idea of proof</u> is finding "rational" directions with "smallest" <u>new</u> "projection" of $M$.

* Here we use the partial ordering of divisibility: $a \precsim b$ if $a \mid b$.
* "rational projection": $D$-mod. homomorphism $\phi: D^n \to D$.
* <u>new</u>: we need certain iterative process; and each time go to a submod. of smaller rank.

<u>Proof</u>. For $\phi \in \mathrm{Hom}_D(D^n, D)$, $\phi(M)$ is a submod. of $D$. That means it is an ideal. Since $D$ is a PID, $\exists\, a_\phi \in D$ st. $\phi(M) = \langle a_\phi \rangle$. Let $\Sigma := \{\langle a_\phi \rangle \mid \phi \in \mathrm{Hom}_D(D^n, D)\}$.

Since $D$ is a PID, it is Noetherian. Hence $\Sigma$ has a maximal element. Suppose $\langle a_1\rangle$ is a maximal element of $\Sigma$; and $y_1 \in M$ and $\phi_1 \in \mathrm{Hom}_D(D^n, D)$ are such that $\phi_1(y_1) = a_1$.

<u>Claim 1</u>. $\forall \phi \in \mathrm{Hom}_D(D^n, D)$, $a_1 \mid \phi(y_1)$.

<u>Pf of claim 1</u>. Since $D$ is a PID, $\exists d \in D$ st. $\langle a_1, \phi(y_1)\rangle = \langle d\rangle$.

So $d = c_1 a_1 + c_2 \phi(y_1) = c_1 \phi_1(y_1) + c_2 \phi(y_1)$.

Let $\psi: D^n \to D$, $\psi(v) := c_1 \phi_1(v) + c_2 \phi(v)$. Then $\psi \in \mathrm{Hom}_D(D^n, D)$ and $\psi(y_1) = d$. Hence $\langle a_1\rangle \subseteq \langle d\rangle \subseteq \psi(M) = \langle a_\psi\rangle$. (I)

Since $\langle a_1\rangle$ is maximal in $\Sigma$, (I) implies $\langle a_1\rangle = \langle a_\psi\rangle$.

And so $\langle a_1\rangle = \langle d\rangle$. Therefore $\phi(y_1) \in \langle a_1\rangle$; this means

$$a_1 \mid \phi(y_1). \qquad \blacksquare$$

<u>Conseq. of Claim 1.</u> Let $pr_i : D^n \to D$ be the proj. to the $i^{th}$ component. Then, by claim 1, $pr_i(y_1) = a_1 c_i$. Hence

$$y_1 = a_1 x_1 \quad \text{where} \quad x_1 = (c_1, c_2, \ldots, c_n).$$

And so $\phi_1(y_1) = a_1 \phi(x_1)$, which implies $\phi(x_1) = 1$.

# Lecture 10: Submodules of free modules over a PID

Friday, February 2, 2018 12:56 AM

<u>Claim 2.</u> $D^n = \ker \phi_1 \oplus D x_1$ and

$$M = (\ker \phi_1 \cap M) \oplus D a_1 x_1 .$$

<u>Pf of claim 2.</u> First we show $D^n = \ker \phi_1 + D x_1$.

(Whenever you want to show a set $T$ is a transverse of the kernel of $\phi$, you have to show $\operatorname{Im}\phi = \phi(T)$.)

$\forall v \in D^n$, $\phi_1(v) = \phi_1(v)\,\phi_1(x_1)$ since $\phi_1(x_1) = 1$

$= \phi_1(\phi_1(v)\, x_1)$ $\phi_1$ is $D$-linear

$\Rightarrow \phi_1(v - \phi_1(v)\, x_1) = 0$

$\Rightarrow v - \phi_1(v)\, x_1 \in \ker \phi_1$

$\Rightarrow v \in \ker \phi_1 + D x_1$.

Second. We show $\ker \phi_1 + D x_1$ is an internal direct sum.

Suppose $\omega + c x_1 = 0 \ (\mathrm{II})$ for $\omega \in \ker \phi_1$ and $c \in D$. Then

$0 = \phi_1(\omega + c x_1) = \underbrace{\phi_1(\omega)}_{0} + c\,\underbrace{\phi_1(x_1)}_{1} = c$. Hence $c = 0$.

And so by (II) $\omega = 0$; and the second subclaim follows.

third. Similar to the first step we show $M = (M \cap \ker \phi_1) + D a_1 x_1$

# Lecture 10: Submodules of free modules over a PID

Friday, February 2, 2018 1:09 AM

$\forall \omega \in M$, $\phi_1(\omega) \in \langle a_1 \rangle \Rightarrow \exists\, c \in D$, $\phi_1(\omega) = c a_1$

$\Rightarrow \phi_1(\omega) = c a_1 \phi_1(x_1) \Rightarrow \omega - c a_1 x_1 \in \ker \phi_1$

Since $\omega$ and $a_1 x_1 = y_1$ are in $M$, $\omega - c a_1 x_1 \in M$.

Hence $\omega \in (\ker \phi_1 \cap M) + D a_1 x_1$; and the subclaim follows.

Fourth We discuss why $(\ker \phi_1 \cap M) + D a_1 x_1$ is an internal direct sum.

$$\left.\begin{array}{l} \ker \phi_1 \cap M \subseteq \ker \phi_1 \\ D a_1 x_1 \subseteq D x_1 \\ \ker \phi_1 + D x_1 \text{ is a direct sum} \end{array}\right\} \Rightarrow \text{ imply the last subclaim.}$$

$\blacksquare$

In the next lecture we will use induction on rank$(M)$ to show $M$ is free; then we will use induction on $n$ to get part (b) of theorem.