Lecture 14: Jordan form

Wednesday, February 7, 2018 11:50 AM

Let k be a field and $A \in M_n(k)$. As before V_A is the $k[x]$-mod k^n where $f(x) \cdot v := f(A)v$.

Assuming the characteristic polynomial $f_A(x)$ of A can be decomposed to linear factors $f_A(x) = \prod (x-\lambda_i)^{n_i}$, we have that the invariant factors $g_j(x) = \prod_i (x-\lambda_i)^{m_{ij}}$. And so by Chinese Remainder Theorem,

$$V_A \simeq \bigoplus_j k[x]/\langle g_j \rangle \simeq \bigoplus_j \bigoplus_i k[x]/\langle (x-\lambda_i)^{m_{ij}} \rangle . \quad (\mathrm{I})$$

Hence we need to find a matrix representation of multiplication by x in $k[x]/\langle (x-\lambda)^m \rangle$.

Let $\theta : k[x]/\langle (x-\lambda)^m \rangle \xrightarrow{\sim} k[y]/\langle y^m \rangle$, $\theta(x) = y+\lambda$.

Then multiplication by y is given by $c(y^m) = \begin{bmatrix} 0 & \cdots & & 0 \\ 1 & & & \vdots \\ & \ddots & & \\ & & 1 & 0 \end{bmatrix}$.

Hence multiplication by x is given by $\lambda I + c(y^m)$ which is $\begin{bmatrix} \lambda & & \\ 1 & \ddots & \\ & \ddots & \\ & 1 & \lambda \end{bmatrix}_{m \times m}$. This is called a Jordan block of size m, and we denote it by $\underline{J_m(\lambda)}$.

Lecture 14: Jordan form

Thursday, February 8, 2018 9:57 PM

Hence (I) implies $V_A \simeq \bigoplus_j \bigoplus_i V_{J_{m_{ij}}(\lambda_i)}$. And so

<u>Theorem (Jordan form)</u> Suppose the characteristic polynomial of A is equal to $\prod_i (x-\lambda_i)^{n_i}$, $(\lambda_i \neq \lambda_j)$. Then A is similar to

(II) $\mathrm{diag}(J_{m_{11}}(\lambda_1), J_{m_{21}}(\lambda_1), \ldots ; J_{m_{12}}(\lambda_2), J_{m_{22}}(\lambda_2), \ldots ; \ldots)$

where $J_{m_{ij}}(\lambda_j) = \begin{bmatrix} \lambda_j & & \\ 1 & \ddots & \\ & \ddots & \\ & 1 & \lambda_j \end{bmatrix}$ and $m_{1j} \leq m_{2j} \leq \cdots$

for any j. (II) is called a Jordan form of A; and it is unique.

<u>Pf.</u> We have already proved the existence of a Jordan form.

Now we briefly discuss why it is unique:

Suppose $A \sim \mathrm{diag}(J_{n_{11}}(\mu_1), \ldots, J_{n_{21}}(\mu_1), \ldots)$. Then by comparing the characteristic polynomials, we get that μ_i's are a reordering of λ_i's. And after reindexing, we can and will assume $\lambda_i = \mu_i$.

Hence $\bigoplus_{i,j} V_{J_{n_{ij}}(\lambda_j)} \simeq \bigoplus_{i,j} V_{J_{m_{ij}}(\lambda_j)}$; and so

$$\bigoplus_{i,j} k[x]/(x-\lambda_j)^{n_{ij}} \simeq \bigoplus_{i,j} k[x]/(x-\lambda_j)^{m_{ij}}. \quad \text{(III)}$$

The rest of argument is similar to the uniqueness of Rational Forms.

Lecture 14: Jordan form

Wednesday, February 7, 2018 8:39 AM

We look at the module of fractions by localizing at the prime ideal $x-\lambda_j$. So let $D_j := k[x]_{\langle x-\lambda_j\rangle}$ and $P_j := x-\lambda_j$.

Then (III) implies, for any j,

$$M := \bigoplus_i D_j/D_j P_j^{m_{ij}} \simeq \bigoplus_i D_j/D_j P_j^{n_{ij}}. \qquad (IV)$$

Now as before using (IV) we can see that the dual of Young Tableau associated to

$$\dim_{D/\langle P\rangle} M/PM \geq \dim_{D/\langle P\rangle} PM/P^2M \geq \cdots$$

gives us both $m_{1j} \leq m_{2j} \leq \cdots$ and $n_{1j} \leq n_{2j} \leq \cdots$

And so $m_{ij} = n_{ij}$. ∎

<u>Corollary</u>. $A \in M_n(k)$ is diagonalizable if and only if its minimal polynomial has distinct zeros.

<u>Pf</u>. ($\Rightarrow$) Since A is diagonalizable, we have $A \sim \mathrm{diag}(\lambda_1 I, \ldots, \lambda_m I)$ where $\lambda_i \neq \lambda_j$. Let $p(x) := \prod_{i=1}^{m} (x-\lambda_i)$. Then

$p(A) \sim p(\mathrm{diag}(\lambda_1 I, \ldots, \lambda_m I)) = \mathrm{diag}(p(\lambda_1) I, \ldots, p(\lambda_m) I) = 0$.

$\Rightarrow p(A) = 0 \Rightarrow m_A(x) \mid p(x) \Rightarrow m_A$ has distinct roots.

Lecture 14: Diagonalizable

Thursday, February 8, 2018 10:38 PM

($\Leftarrow$) Suppose $g_1 | g_2 | \cdots | g_k$ are the invariant factors, and

$m_A(x) = \prod_{i=1}^{\ell} (x-\lambda_i)$ where $\lambda_i \neq \lambda_j$. Then, since

$g_k(x) = m_A(x)$, we get that

$\exists\ S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq \{1, 2, \ldots, \ell\}$ such that

$$g_j(x) = \prod_{i \in S_j} (x - \lambda_i).$$

Hence
$$V_A \simeq \bigoplus_j k[x]/\langle g_j \rangle \simeq \bigoplus_j \bigoplus_{i \in S_j} \underbrace{k[x]/\langle x-\lambda_i \rangle}_{V_{[\lambda_i]}}$$

$$\simeq V_{\mathrm{diag}(\lambda_1 I_{m_1}, \lambda_2 I_{m_2}, \ldots, \lambda_\ell I_{m_\ell})}$$

where $m_i = |\{ j \in [1..k] \mid i \in S_j \}|$. And so A is diagonalizable.

■

Now that we have seen how important and instrumental module theory is, we try to study them a bit more systematically.

Lecture 14: Simple modules

Thursday, February 8, 2018 11:09 PM

As in group theory, we can start with "simplest" R-modules and try to build all the modules out of them.

Def. We say M is a simple R-module (or an irreducible R-module) if 0 and M are its only submodules and $M \neq 0$.

Lemma (a) Suppose M_1 and M_2 are two simple R-mod. Then

$$\mathrm{Hom}_R(M_1, M_2) \neq 0 \iff M_1 \simeq M_2$$

(b) (Schur's lemma) Suppose M is a simple R-mod. Then $\mathrm{End}_R M$ is a division ring.

Pf. (a) ($\Leftarrow$) is clear.

($\Rightarrow$) Let $\phi: M_1 \to M_2$ be a non-zero R-mod hom. Then $\ker \phi$ is a proper submod. Since M_1 is simple, we deduce that $\ker \phi = 0$.

Since ϕ is non-zero, $\mathrm{Im}\, \phi$ is not zero. Since M_2 is simple $\mathrm{Im}\, \phi = M_2$. Hence ϕ is injective and surjective. Therefore it is an isomorphism, and $M_1 \simeq M_2$.

(b) Let $\phi \in \mathrm{End}_R(M)$ and $\phi \neq 0$. By the argument of part (a) we have that ϕ is an isomorphism, and so $\phi^{-1} \in \mathrm{End}_R M$. ∎

Lecture 14: Exact sequences

Friday, February 9, 2018 8:30 AM

As in group theory, we use exact sequences in order to split a problem about modules into easier pieces.

<u>Def</u> (a) We say $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n$ is an <u>exact sequence</u> if $f_i \in \mathrm{Hom}_R(M_i, M_{i+1})$ and

$$\mathrm{im}\, f_i = \ker f_{i+1};$$

in particular $f_{i+1} \circ f_i = 0$.

(b) An exact sequence of the form

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

is called a <u>short exact sequence</u>.

<u>Ex</u> (a) $0 \to M_1 \xrightarrow{f_1} M_2$ is an exact sequence

$$\Updownarrow$$

f_1 is injective.

(b) $M_1 \xrightarrow{f_1} M_2 \to 0$ is an exact sequence

$$\Updownarrow$$

f_1 is surjective.

(c) If $0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$ is a short exact sequence, then

Lecture 14: Short exact sequences

Friday, February 9, 2018 8:40 AM

there are isomorphisms Φ_1, Φ_2, and Φ_3 such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "f_1"] \arrow[d, "\Phi_1"'] \arrow[dr, phantom, "\circlearrowleft"] & M_2 \arrow[r, "f_2"] \arrow[d, "\Phi_2"'] \arrow[dr, phantom, "\circlearrowleft"] & M_3 \arrow[r] \arrow[d, "\Phi_3"'] & 0 \\
0 \arrow[r] & f_1(M_1) \arrow[r] & M_2 \arrow[r] & M_2/f_1(M_1) \arrow[r] & 0
\end{tikzcd}$$

<u>Pf.</u> f_1 is injective and f_2 is surjective.

And so $f_1 : M_1 \to f_1(M_1)$ is an isomorphism and

$\overline{f_2} : M_2/\ker f_2 \longrightarrow M_3$, $\overline{f_2}(x_2 + \ker f_2) := f_2(x_2)$

is an isomorphism.

Let $\Phi_1 : M_1 \to f_1(M_1)$, $\Phi_1(x_1) := f_1(x_1)$,

$\Phi_2 := \mathrm{id}_{M_2}$, and

$\Phi_3 := \overline{f_2}^{-1} : M_3 \xrightarrow{\sim} M_2/\ker f_2 = M_2/f_1(M_1)$.

Then one can easily check that the above diagram commutes. ∎