Lecture 16: Representable functors

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In the previous lecture we defined <u>functor</u>. Here is an important example

<u>Ex.</u> Suppose $\mathrm{Hom}_{\mathcal{C}}(a,b)$ is a set for any $a,b\in \mathrm{Obj}(\mathcal{C})$.

For a given $a_0\in\mathrm{Obj}(\mathcal{C})$, let

$$F_{a_0}:\mathcal{C}\to \mathrm{Set},\ F_{a_0}(b):=\mathrm{Hom}_{\mathcal{C}}(a_0,b)$$

and, for $\phi\in\mathrm{Hom}_{\mathcal{C}}(b_1,b_2)$, let

$$F_{a_0}(\phi): F_{a_0}(b_1)\to F_{a_0}(b_2),\ F_{a_0}(\phi)(\psi):=\phi\circ\psi$$

$$a_0\xrightarrow{\psi} b_1\xrightarrow{\phi} b_2,\quad a_0\xrightarrow{\phi\circ\psi} b_2$$

Then F_{a_0} is a functor.

- $a_0\xrightarrow{\psi} b\xrightarrow{id_b} b$, $a_0\xrightarrow{\psi} b$ so $F_{a_0}(id_b)=id_{F_{a_0}(b)}$
- $a_0\to b_1\xrightarrow{\phi_1} b_2\xrightarrow{\phi_2} b_3$, $b_1\xrightarrow{\phi_2\circ\phi_1} b_3$

$$\begin{aligned}F_{a_0}(\phi_2\circ\phi_1)(\psi)&=\phi_2\circ\phi_1\circ\psi\\ &=F_{a_0}(\phi_2)(\phi_1\circ\psi)\\ &=F_{a_0}(\phi_2)(F_{a_0}(\phi_1)(\psi))\\ &=(F_{a_0}(\phi_2)\circ F_{a_0}(\phi_1))(\psi).\end{aligned}$$

Roughly this type of functors are called <u>representable</u>.

In the category of R-mod, let's look at a representable functor. We notice that $F_M(N):=\mathrm{Hom}_R(M,N)$ is an abelian gp. So first we see whether $F_M: R\text{-mod}\to \mathrm{Ab}$.

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Lemma. For $\phi \in \mathrm{Hom}_R(N_1, N_2)$, $F_M(\phi) \in \mathrm{Hom}(F_M(N_1), F_M(N_2))$ (where F_M is the $\mathrm{Hom}_R(M, -)$ functor)

Pf. $F_M(\phi)(\psi_1 - \psi_2) = \phi \circ (\psi_1 - \psi_2) = \phi \circ \psi_1 - \phi \circ \psi_2$

$$= F_M(\phi)(\psi_1) - F_M(\phi)(\psi_2). \quad \blacksquare$$

Next we will investigate whether injective or surjective maps are sent to injective or surjective, resp.

Lemma. If $\phi \in \mathrm{Hom}_R(N_1, N_2)$ is injective, then $F_M(\phi)$ is injective.

$$\begin{array}{ccc} R\text{-mod} & & Ab \\ N_1 & & \mathrm{Hom}_R(M, N_1) \\ \downarrow \phi & & \downarrow F_M(\phi) =: \hat{\phi} \\ N_2 & & \mathrm{Hom}_R(M, N_2) \end{array}$$

$$M \xrightarrow{\psi} N_1 \xrightarrow{\phi} N_2, \qquad M \xrightarrow{(F_M(\phi))(\psi)} N_2$$

Pf. Suppose $(F_M(\phi))(\psi) = 0$.

Then $\phi(\psi(x)) = 0 \ \forall x \in M$. As ϕ is injective, $\psi(x) = 0 \ \forall x$.

Hence $\psi = 0$. Thus $F_M(\phi)$ is injective. $\blacksquare$

Theorem. If $0 \to N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3 \to 0$ is a short exact seq., then

$$0 \to \mathrm{Hom}_R(M, N_1) \xrightarrow{\hat{f}_1} \mathrm{Hom}_R(M, N_2) \xrightarrow{\hat{f}_2} \mathrm{Hom}_R(M, N_3)$$

is exact. (But $\hat{f}_2$ is NOT necessarily surjective).

Pf. We have already proved that $\hat{f}_1$ is injective and

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$\hat{f}_2 \circ \hat{f}_1 = F_M(f_2) \circ F_M(f_1) = F_M(\underbrace{f_2 \circ f_1}_{0}) = 0$. And so

$\operatorname{Im} \hat{f}_1 \subseteq \ker \hat{f}_2$. So it is enough to show $\ker \hat{f}_2 \subseteq \operatorname{Im} \hat{f}_1$.

Suppose $\phi \in \ker \hat{f}_2$.

$$\begin{tikzcd} & & M \arrow[d, "\phi"] & & \\ 0 \arrow[r] & N_1 \arrow[r, "f_1"] & N_2 \arrow[r, "f_2"] & N_3 \arrow[r] & 0 \end{tikzcd}$$

So $f_2 \circ \phi = 0$. Hence

$\operatorname{Im} \phi \subseteq \ker f_2 = \operatorname{Im} f_1$. $\Rightarrow \forall x \in M, \exists y_1 \in N_1$ s.t. $\phi(x) = f_1(y_1)$

Since f_1 is injective, $\exists! \, y_1 \in N_1$ s.t. $\phi(x) = f_1(y_1)$.

Let $\Phi : M \to N_1$, $\Phi(x) := y_1$.

$$\begin{tikzcd} & & M \arrow[d, "\phi"] \arrow[dl, dashed, "\Phi"'] & & \\ 0 \arrow[r] & N_1 \arrow[r, "f_1"] & N_2 \arrow[r, "f_2"] & N_3 \arrow[r] & 0 \end{tikzcd}$$

If $\Phi(x) = y_1$ and $\Phi(x') = y_1'$, then

$$f_1(y_1 + r y_1') = f_1(y_1) + r f_1(y_1') = \phi(x) + r\phi(x')$$

$$= \phi(x + r x') \Rightarrow \Phi(x + r x') = y_1 + r y_1'$$

$$= \Phi(x) + r\Phi(x').$$

And so $\Phi \in \operatorname{Hom}_R(M, N_1)$ and $\phi = f_1 \circ \Phi$

$$= F_M(f_1)(\Phi)$$

$\Rightarrow \phi \in \operatorname{Im} \hat{f}_1$.

Notice that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$; And so

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$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ and

$$0 \to \underset{\substack{\| \\ 0}}{\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})} \to \underset{\substack{\| \\ 0}}{\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})} \to \underset{\substack{\not\| \\ 0}}{\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})} \not\to 0 \quad \blacksquare$$

<u>Theorem</u>. The following statements are equivalent:

(a) $\mathrm{Hom}_R(P, \_)$ functor is an exact functor; that means

if $0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$ is a S.E.S., then

$$0 \to \mathrm{Hom}_R(P, M_1) \xrightarrow{\hat{f}_1} \mathrm{Hom}_R(P, M_2) \xrightarrow{\hat{f}_2} \mathrm{Hom}_R(P, M_3) \to 0 \text{ is a S.E.S.}$$

(b) If ϕ is surjective, then $\hat{\phi}$ is surjective.

(c) If $M \xrightarrow{\phi} N \to 0$ is exact, then any $f \in \mathrm{Hom}_R(P, N)$ has a lift $F \in \mathrm{Hom}_R(P, M)$; that means

$$\begin{tikzcd} & P \arrow[dl, dashed, "F"'] \arrow[d, "f"] \arrow[dl, phantom, shift left=3, "\circlearrowleft"] & \\ M \arrow[r, "\phi"'] & N \arrow[r] & 0 \end{tikzcd}$$

(d) Any S.E.S. of the form $0 \to M_1 \to M_2 \to P \to 0$ splits.

(e) P is a direct summand of a free R-mod; that means

$\exists$ an R-mod N and a free R-mod F s.t. $F \simeq P \oplus N$.

<u>Pf.</u> (a) $\Leftrightarrow$ (b) Previous Proposition gives us this.

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(b) $\Rightarrow$ (c) Since ϕ is surjective, $\hat{\phi}$ is surjective. So $\exists\, F \in \mathrm{Hom}_R(P, M_1)$ s.t. $\hat{\phi}(F) = f$; that means $\phi \circ F = f$, which means F is a lift of f.

$$\begin{tikzcd} & P \arrow[dl, dashed, "F"'] \arrow[d, "f"] & \\ M_1 \arrow[r, "\phi"'] & M_2 \arrow[r] & 0 \end{tikzcd}$$

(c) $\Rightarrow$ (b) Suppose $\phi : M_1 \to M_2$ is surjective, and $f \in \mathrm{Hom}_R(P, M_2)$. Then $\exists\, F \in \mathrm{Hom}_R(P, M_1)$, $\hat{\phi}(F) = f$; this means

$$\begin{tikzcd} & P \arrow[dl, dashed, "F"'] \arrow[d, "f"] & \\ M_1 \arrow[r, "\phi"'] & M_2 \arrow[r] & 0 \end{tikzcd}$$

(c) $\Rightarrow$ (d) Suppose $0 \to M_1 \xrightarrow[f_1]{} M_2 \xrightarrow[f_2]{} P \to 0$ is a S.E.S..

Consider $f := \mathrm{id}_P : P \to P$. By (b), id_P has a lift $g_2 \in \mathrm{Hom}_R(P, M_2)$; that means

$$\begin{tikzcd} & & & P \arrow[dl, dashed, "g_2"'] \arrow[d, equal] & \\ 0 \arrow[r] & M_1 \arrow[r, "f_1"'] & M_2 \arrow[r, "f_2"'] & P \arrow[r] & 0 \end{tikzcd}$$

And so this S.E.S. splits.

(d) $\Rightarrow$ (e) Let $F(P)$ be the free R-mod. generated by the set P. By the universal property of free modules $\exists\, \phi : F(P) \to P$, s.t. $\phi(x) = x$ $\forall x \in P$; in parti.

$$0 \to \ker \phi \to F(P) \to P \to 0$$

is a S.E.S. Hence $F(P) \simeq \ker \phi \oplus P$.

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(e) $\Rightarrow$ (c) Suppose $F(A) \simeq P \oplus N$ where F is a free R-mod.

$$\begin{tikzcd} & F(A) \simeq P \oplus N \arrow[d, "pr"] & \\ & P \arrow[d, "f"] & \\ M_1 \arrow[r, "\phi"'] & M_2 \arrow[r] & 0 \end{tikzcd}$$

For any $a \in A$, let $x_a \in M_1$ be such that

$$\phi(x_a) = f(pr(a)).$$

By the universal property of free R-modules,

$\exists!\ \tilde{F} \in \mathrm{Hom}_R(F(A), M_1)$, $\tilde{F}(a) := x_a$. So the following diagram commutes:

$$\begin{tikzcd} & A \arrow[dl, "\tilde{F}|_A"'] \arrow[d, "f \circ pr|_A"] \\ M_1 \arrow[r] & M_2 \end{tikzcd}$$

As $F(A)$ is generated by A, we get that the following is a commuting diagram:

$$\begin{tikzcd} & F(A) \simeq P \oplus N \arrow[dl, "\tilde{F}"'] \arrow[d, shift left, "pr"] & \\ & P \arrow[u, shift left, "j"] \arrow[d, "f"] \arrow[dl, dashed, "F"'] & \\ M_1 \arrow[r] & M_2 \arrow[r] & 0 \end{tikzcd}$$

Let $F: P \to M_1$, $F(x) := \tilde{F} \circ j(x)$
$= \tilde{F}(x, 0)$

So one can see that F is a lift of f. $\blacksquare$

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<u>Def</u>. A module which satisfies the equivalent properties in the previous theorem is called a projective module.

<u>Corollary</u>. Suppose D is an integral domain. Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c)

(a) M is free. (b) M is projective. (c) M is torsion free.

Moreover, if D is a PID and M is f.g., then (a), (b), and (c) are equivalent.

<u>Pf</u>. (a) $\Rightarrow$ (b) ✓ (M is a direct summand of a free mod!)

(b) $\Rightarrow$ (c) M is a direct summand of a free mod.

$$\Rightarrow M \hookrightarrow \bigoplus_{i\in I} D.$$

Suppose $(x_i) \in \mathrm{Tor}(M)$. So $\exists\, d \in D\setminus\{0\}$ s.t. $d(x_i) = 0$

$$\Rightarrow \forall i,\ \left.\begin{array}{l} d\,x_i = 0 \\ d \neq 0 \end{array}\right\} \Rightarrow x_i = 0 \Rightarrow (x_i) = 0.$$

. When D is a PID and M is a torsion-free f.g. D-mod, then by the classification of such modules M is free. ∎

<u>Ex.</u> $\langle 2, \sqrt{-10} \rangle$ is not a free $\mathbb{Z}[\sqrt{-10}]$-module, but it is projective.

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Solution. Show I is not a principal ideal; but, as $I \subseteq R$

(where $R = \mathbb{Z}[\sqrt{-10}]$), rank $I \leq 1$.

Consider
$$0 \to \ker\phi \to R^2 \to I \to 0$$
$$(x,y) \longmapsto 2x + \sqrt{-10}\, y$$

If we show it splits, then $R^2 \simeq I \oplus \ker\phi$; and so I is

projective. Hence we need to find

$$\psi: I \to R^2, \quad \psi(x) := (c_1 x, c_2 x)$$

s.t ① $c_1, c_2 \in$ field of fraction of $R = \mathbb{Q}[\sqrt{-10}]$

② $c_i I \subseteq R$

③ $2c_1 x + \sqrt{-10}\, c_2 x = x$ which is equivalent to

$2c_1 + \sqrt{-10}\, c_2 = 1$.

Let $c_i = x_i + \sqrt{-10}\, y_i \in \mathbb{Q}[\sqrt{-10}]$.

$$\Rightarrow \left.\begin{cases} 2c_i \in \mathbb{Z}[\sqrt{-10}] \iff x_i, y_i \in \frac{1}{2}\mathbb{Z} \\ \sqrt{-10}\, c_i \in \mathbb{Z}[\sqrt{-10}] \iff x_i \in \mathbb{Z},\ y_i \in \frac{1}{10}\mathbb{Z} \end{cases}\right\} \iff x_i \in \mathbb{Z},\ y_i \in \tfrac{1}{2}\mathbb{Z}.$$

and $2x_1 + \sqrt{-10}\, 2y_1 + \sqrt{-10}\, x_2 - 10 y_2 = 1 \iff \begin{cases} 2x_1 - 10y_2 = 1 \\ 2y_1 + x_2 = 0 \end{cases}$

So $x_1 = 3$, $y_2 = \frac{1}{2}$, $x_2 = y_1 = 0$ work:

$$\psi: I \to R^2, \quad \psi(x) = \left(3x, \frac{\sqrt{-10}}{2} x\right).$$