

Lecture 16: Representable functors

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In the previous lecture we defined functor. Here is an important example

Ex. Suppose $\text{Hom}_{\mathcal{C}}(a, b)$ is a set for any $a, b \in \text{Obj}(\mathcal{C})$.

For a given $a_0 \in \text{Obj}(\mathcal{C})$, let

$$F_{a_0} : \mathcal{C} \rightarrow \text{Set}, \quad F_{a_0}(b) := \text{Hom}_{\mathcal{C}}(a_0, b)$$

and, for $\phi \in \text{Hom}_{\mathcal{C}}(b_1, b_2)$, let

$$F_{a_0}(\phi) : F_{a_0}(b_1) \rightarrow F_{a_0}(b_2), \quad F_{a_0}(\phi)(\varphi) := \phi \circ \varphi$$

$$\begin{array}{ccccc} a_0 & \xrightarrow{\varphi} & b_1 & \xrightarrow{\phi} & b_2 \\ & & \searrow & \xrightarrow{\phi \circ \varphi} & \\ & & & & \end{array}$$

Then F_{a_0} is a functor.

$$\bullet \quad \begin{array}{ccc} a_0 & \xrightarrow{\varphi} & b \xrightarrow{\text{id}_b} b \\ & \xrightarrow{\varphi} & \end{array} \quad \text{so } F_{a_0}(\text{id}_b) = \text{id}_{F_{a_0}(b)}$$

$$\bullet \quad \begin{array}{ccc} a_0 & \rightarrow & b_1 \xrightarrow{\phi_1} b_2 \xrightarrow{\phi_2} b_3 \\ & & \xrightarrow{\phi_2 \circ \phi_1} \end{array} \quad \begin{aligned} F_{a_0}(\phi_2 \circ \phi_1)(\varphi) &= \phi_2 \circ \phi_1 \circ \varphi \\ &= F_{a_0}(\phi_2)(\phi_1 \circ \varphi) \\ &= F_{a_0}(\phi_2)(F_{a_0}(\phi_1)(\varphi)) \\ &= (F_{a_0}(\phi_2) \circ F_{a_0}(\phi_1))(\varphi). \end{aligned}$$

Roughly this type of functors

are called representable.

In the category of \mathbb{R} -mod, let's look at a representable functor. We notice that $F_M(N) := \text{Hom}_{\mathbb{R}}(M, N)$ is an abelian gp. So first we see whether $F_M : \mathbb{R}\text{-mod} \rightarrow \text{Ab}$.

Lecture 16: Representable functors of R-mod

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Lemma. For $\phi \in \text{Hom}_R(N_1, N_2)$, $F_M(\phi) \in \text{Hom}(F_M(N_1), F_M(N_2))$

(where F_M is the $\text{Hom}_R(M, -)$ functor)

Pf. $F_M(\phi)(\psi_1 - \psi_2) = \phi \circ (\psi_1 - \psi_2) = \phi \circ \psi_1 - \phi \circ \psi_2$
 $= F_M(\phi)(\psi_1) - F_M(\phi)(\psi_2). \quad \blacksquare$

Next we will investigate whether injective or surjective maps are sent to injective or surjective, resp.

Lemma. If $\phi \in \text{Hom}_R(N_1, N_2)$ is injective, then $F_M(\phi)$ is injective.

$R\text{-mod}$	Ab
N_1	$\text{Hom}_R(M, N_1)$
$\downarrow \phi$	$\downarrow F_M(\phi) =: \hat{\phi}$
N_2	$\text{Hom}_R(M, N_2)$

Pf. Suppose $(F_M(\phi))(\psi) = 0$.

$$M \xrightarrow{\psi} N_1 \xrightarrow{\phi} N_2$$

$\xrightarrow{(F_M(\phi))(\psi)}$

Then $\phi(\psi(x)) = 0 \quad \forall x \in M$. As ϕ is injective, $\psi(x) = 0 \quad \forall x$.

Hence $\psi = 0$. Thus $F_M(\phi)$ is injective. \blacksquare

Theorem. If $0 \rightarrow N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3 \rightarrow 0$ is a short exact seq.,

then $0 \rightarrow \text{Hom}_R(M, N_1) \xrightarrow{\hat{f}_1} \text{Hom}_R(M, N_2) \xrightarrow{\hat{f}_2} \text{Hom}_R(M, N_3)$

is exact. (But \hat{f}_2 is NOT necessarily surjective).

Pf. We have already proved that \hat{f}_1 is injective and

Lecture 16: Left exactness of $\text{Hom}(M, _)$

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$$\hat{f}_2 \circ \hat{f}_1 = F_M(f_2) \circ F_M(f_1) = F_M(\underbrace{f_2 \circ f_1}_0) = 0. \text{ And so}$$

$\text{Im } \hat{f}_1 \subseteq \ker \hat{f}_2$. So it is enough to show $\ker \hat{f}_2 \subseteq \text{Im } \hat{f}_1$.

Suppose $\phi \in \ker \hat{f}_2$.

$$\begin{array}{ccccccc} & & & M & & & \\ & & & \downarrow \phi & & & \\ 0 & \rightarrow & N_1 & \xrightarrow{f_1} & N_2 & \xrightarrow{f_2} & N_3 \rightarrow 0 \end{array}$$

So $f_2 \circ \phi = 0$. Hence

$\text{Im } \phi \subseteq \ker f_2 = \text{Im } f_1 \Rightarrow \forall x \in M, \exists y_1 \in N_1 \text{ s.t.}$

Since f_1 is injective, $\exists! y_1 \in N_1 \text{ s.t.}$

$$\phi(x) = f_1(y_1)$$

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Let $\bar{\phi} : M \rightarrow N_1, \bar{\phi}(x) := y_1$.

$$\begin{array}{ccccccc} & & & M & & & \\ & & & \downarrow \phi & & & \\ & & \bar{\phi} & \nearrow & & & \\ 0 & \rightarrow & N_1 & \xrightarrow{f_1} & N_2 & \xrightarrow{f_2} & N_3 \rightarrow 0 \end{array}$$

If $\bar{\phi}(x) = y_1$ and $\bar{\phi}(x') = y_1'$, then

$$f_1(y_1 + r y_1') = f_1(y_1) + r f_1(y_1') = \phi(x) + r \phi(x')$$

$$= \phi(x + r x') \Rightarrow \bar{\phi}(x + r x') = y_1 + r y_1'$$

$$= \bar{\phi}(x) + r \bar{\phi}(x').$$

And so $\bar{\phi} \in \text{Hom}_{\mathbb{R}}(M, N_1)$ and $\phi = f_1 \circ \bar{\phi}$

$$= F_M(f_1)(\bar{\phi})$$

$\Rightarrow \phi \in \text{Im } \hat{f}_1$.

Notice that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$; And so

Lecture 16: Not necessarily an exact functor

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$$0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow \text{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \rightarrow \text{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \rightarrow \text{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \rightarrow 0$$

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Theorem. The following statements are equivalent:

(a) $\text{Hom}_{\mathbb{R}}(\mathbb{P}, -)$ functor is an exact functor; that means

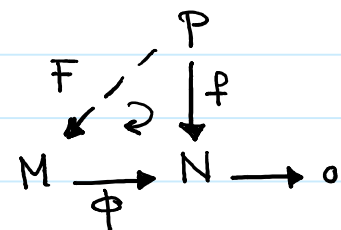
if $0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \rightarrow 0$ is a S.E.S., then

$$0 \rightarrow \text{Hom}_{\mathbb{R}}(\mathbb{P}, M_1) \xrightarrow{\hat{f}_1} \text{Hom}_{\mathbb{R}}(\mathbb{P}, M_2) \xrightarrow{\hat{f}_2} \text{Hom}_{\mathbb{R}}(\mathbb{P}, M_3) \rightarrow 0 \text{ is a S.E.S.}$$

(b) If ϕ is surjective, then $\hat{\phi}$ is surjective.

(c) If $M \xrightarrow{\phi} N \rightarrow 0$ is exact, then any $f \in \text{Hom}_{\mathbb{R}}(\mathbb{P}, N)$ has

a lift $F \in \text{Hom}_{\mathbb{R}}(\mathbb{P}, M)$; that means



(d) Any S.E.S. of the form $0 \rightarrow M_1 \rightarrow M_2 \rightarrow \mathbb{P} \rightarrow 0$ splits.

(e) \mathbb{P} is a direct summand of a free \mathbb{R} -mod; that means

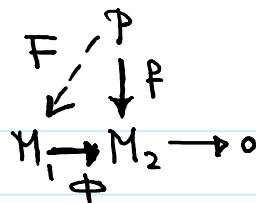
$$\exists \text{ an } \mathbb{R}\text{-mod } N \text{ and a free } \mathbb{R}\text{-mod } F \text{ s.t. } F \simeq \mathbb{P} \oplus N.$$

Pf. (a) \iff (b) Previous Proposition gives us this.

Lecture 16: Projective modules

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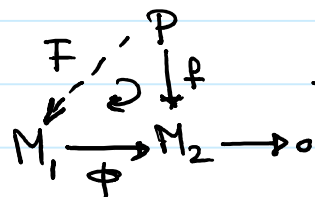
(b) \Rightarrow (c) Since ϕ is surjective, $\hat{\phi}$ is



surjective. So $\exists F \in \text{Hom}_{\mathbb{R}}(P, M_1)$ s.t. $\hat{\phi}(F) = f$; that means $\phi \circ F = f$, which means F is a lift of f .

(c) \Rightarrow (b) Suppose $\phi: M_1 \rightarrow M_2$ is surjective, and $f \in \text{Hom}_{\mathbb{R}}(P, M_2)$.

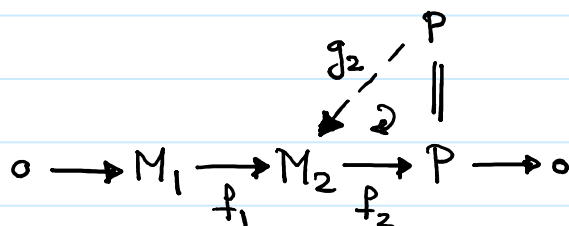
Then $\exists F \in \text{Hom}_{\mathbb{R}}(P, M_1)$, $\hat{\phi}(F) = f$; this means



(c) \Rightarrow (d) Suppose $0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} P \rightarrow 0$ is a S.E.S.

Consider $f := \text{id}_P: P \rightarrow P$. By (b), id_P has a lift $g_2 \in \text{Hom}_{\mathbb{R}}(P, M_2)$:

that means



And so this S.E.S. splits.

(d) \Rightarrow (e) Let $F(\mathcal{I})$ be the free \mathbb{R} -mod. generated by

the set \mathcal{I} . By the universal property of free modules

$\exists \phi: F(\mathcal{I}) \rightarrow P$, s.t. $\phi(x) = x \quad \forall x \in \mathcal{I}$; in parti.

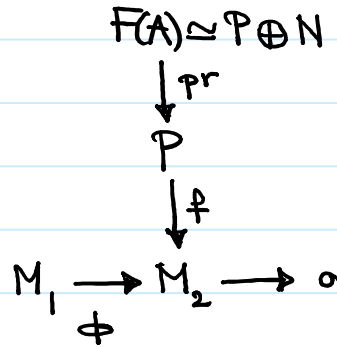
$$0 \rightarrow \ker \phi \rightarrow F(\mathcal{I}) \rightarrow P \rightarrow 0$$

is a S.E.S. Hence $F(\mathcal{I}) \cong \ker \phi \oplus P$.

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(e) \Rightarrow (c) Suppose $F(A) \cong P \oplus N$ where F is a free R -mod.



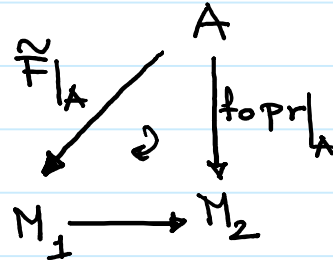
For any $a \in A$, let $x_a \in M_1$ be such that

$$\phi(x_a) = f(\text{pr}(a)).$$

By the universal property of free R -modules,

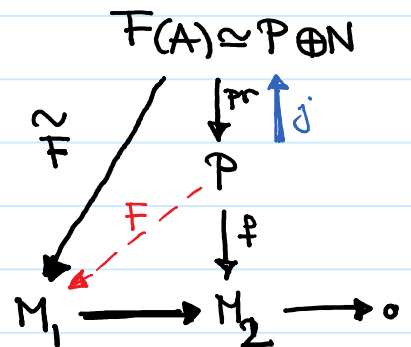
$\exists! \tilde{F} \in \text{Hom}_R(F(A), M_1)$, $\tilde{F}(a) := x_a$. So the following

diagram commutes:



As $F(A)$ is generated by A , we get that the following

is a commuting diagram:



Let $F: P \rightarrow M_1$, $F(x) := \tilde{F} \circ j(x)$
 $= \tilde{F}(x, 0)$

So one can see that F is a lift of f . ■

Lecture 16: Projective modules

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Def. A module which satisfies the equivalent properties in the previous theorem is called a projective module.

Corollary. Suppose D is an integral domain. Then $(a) \Rightarrow (b) \Rightarrow (c)$

(a) M is free. (b) M is projective. (c) M is torsion free.

Moreover, if D is a PID and M is f.g., then (a), (b), and (c) are equivalent.

Pf. (a) \Rightarrow (b) \checkmark (M is a direct summand of a free mod!)

(b) \Rightarrow (c) M is a direct summand of a free mod.

$$\Rightarrow M \hookrightarrow \bigoplus_{i \in I} D.$$

Suppose $(x_i) \in \text{Tor}(M)$. So $\exists d \in D \setminus \{0\}$ s.t. $d(x_i) = 0$

$$\Rightarrow \forall i, \left. \begin{array}{l} d x_i = 0 \\ d \neq 0 \end{array} \right\} \Rightarrow x_i = 0 \Rightarrow (x_i) = 0.$$

. When D is a PID and M is a torsion-free f.g. D -mod,

then by the classification of such modules M is free. ■

Ex. $\langle 2, \sqrt{-10} \rangle$ is not a free $\mathbb{Z}[\sqrt{-10}]$ -module, but it is projective.

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Solution. Show I is not a principal ideal; but, as $I \subseteq R$

(where $R = \mathbb{Z}[\sqrt{-10}]$), $\text{rank } I \leq 1$.

Consider $0 \rightarrow \ker \phi \rightarrow R^2 \rightarrow I \rightarrow 0$
 $(x, y) \mapsto 2x + \sqrt{-10}y$

If we show it splits, then $R^2 \cong I \oplus \ker \phi$; and so I is projective. Hence we need to find

$$\psi: I \rightarrow R^2, \psi(x) := (c_1 x, c_2 x)$$

st ① $c_1, c_2 \in$ field of fraction of $R = \mathbb{Q}[\sqrt{-10}]$

② $c_i I \subseteq R$

③ $2c_1 x + \sqrt{-10}c_2 x = x$ which is equivalent to
 $2c_1 + \sqrt{-10}c_2 = 1$.

Let $c_i = x_i + \sqrt{-10}y_i \in \mathbb{Q}[\sqrt{-10}]$.

$$\Rightarrow \left\{ \begin{array}{l} 2c_i \in \mathbb{Z}[\sqrt{-10}] \iff x_i, y_i \in \frac{1}{2}\mathbb{Z} \\ \sqrt{-10}c_i \in \mathbb{Z}[\sqrt{-10}] \iff x_i \in \mathbb{Z}, y_i \in \frac{1}{10}\mathbb{Z} \end{array} \right\} \iff x_i \in \mathbb{Z}, y_i \in \frac{1}{2}\mathbb{Z}$$

and $2x_1 + \sqrt{-10}2y_1 + \sqrt{-10}x_2 - 10y_2 = 1 \iff \begin{cases} 2x_1 - 10y_2 = 1 \\ 2y_1 + x_2 = 0 \end{cases}$

So $x_1 = 3, y_2 = \frac{1}{2}, x_2 = y_1 = 0$ work:

$$\psi: I \rightarrow R^2, \psi(x) = (3x, \frac{\sqrt{-10}}{2}x)$$