Lecture 18: What we proved in the previous lecture

Tuesday, February 20, 2018 9:00 AM

In the previous lecture we proved the following results:

Suppose M is an (S,R)-bi-module and N is a right S-module.

Then there are a right R-module $N\otimes_S M$ and a function

$f_\circ : N\times M \longrightarrow N\otimes_S M$, $f_\circ(n,m) = n\otimes m$ such that

(1-a) $(n\cdot s)\otimes m = n\otimes(s\cdot m)$ (S-balanced)

(1-b) $(n_1-n_2)\otimes m = n_1\otimes m - n_2\otimes m$ (linear in N)

(1-c) $n\otimes(m_1r_1+m_2r_2) = (n\otimes m_1)r_1 + (n\otimes m_2)r_2$ (R-linear in M)

(2) (Universal property) If L is a right R-module, and $f: N\times M \longrightarrow L$

satisfies properties (1-a)-(1-c), then $\exists!\ \psi: N\otimes_S M \longrightarrow L$

that is an R-mod homomorphism and $\psi(n\otimes m) = f(n,m)$

$$\begin{tikzcd} & N\otimes_S M \arrow[d, dashed, "\psi"] \\ N\times M \arrow[ur, "f_\circ"] \arrow[r, phantom, "\circlearrowleft"] \arrow[dr, "f"'] & \\ & L \end{tikzcd}$$

(3) (Tensor-Hom adjunction)

There is natural transformation between $F_N \circ F_M$ and $F_{N\otimes_S M}$;

that means for $\phi \in \mathrm{Hom}_R(L_1,L_2)$

$$\begin{tikzcd} \mathrm{Hom}_R(N\otimes_S M, L_1) \arrow[r, "\sim", "\eta_{L_1}"'] \arrow[d, "F_{N\otimes_S M}(\phi)"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_S(N, \mathrm{Hom}_R(M,L_1)) \arrow[d, "F_N(F_M(\phi))"] \\ \mathrm{Hom}_R(N\otimes_S M, L_2) \arrow[r, "\sim", "\eta_{L_2}"'] & \mathrm{Hom}_S(N, \mathrm{Hom}_R(M,L_2)) \end{tikzcd}$$

Lecture 18: Tensor product of projective modules

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Corollary. Suppose M is an (S,R)-bi-module and a projective right R-module. Suppose N is a projective right S-module. Then $N \otimes_S M$ is a projective right R-module.

Proof. Since M is projective, F_M is an exact functor. Since N is projective, F_N is an exact functor. Hence $F_N \circ F_M$ is an exact sequence. Let $\eta: F_{N\otimes_S M} \to F$ be a natural transformation. Then for any short exact sequence of right R-modules $0 \to L_1 \xrightarrow{\phi_1} L_2 \xrightarrow{\phi_2} L_3 \to 0$ we get

S.E.S.
$$\begin{tikzcd}
0 \arrow[r] & F_N \circ F_M(L_1) \arrow[r, "F_N\circ F_M(\phi_1)"] \arrow[dr, phantom, "\circlearrowleft"] & F_N \circ F_M(L_2) \arrow[r, "F_N\circ F_M(\phi_2)"] \arrow[dr, phantom, "\circlearrowleft"] & F_N\circ F_M(L_3) \arrow[r] & 0 \\
& F_{N\otimes_S M}(L_1) \arrow[u, "\eta_{L_1}", "\wr"'] \arrow[r, "F_{N\otimes_S M}(\phi_1)"'] & F_{N\otimes_S M}(L_2) \arrow[u, "\eta_{L_2}", "\wr"'] \arrow[r, "F_{N\otimes_S M}"'] & F_{N\otimes_S M}(L_3) \arrow[u, "\eta_{L_3}", "\wr"'] &
\end{tikzcd}$$

Hence the 2nd row is a S.E.S. as well. And so $N\otimes_S M$ is a projective module. ∎

Lecture 18: Examples of tensor product

Wednesday, February 14, 2018 11:26 PM

Ex. $\mathbb{Z}/_{n\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}/_{m\mathbb{Z}} \simeq \mathbb{Z}/_{\gcd(n,m)\mathbb{Z}}$ as abelian groups.

Pf. $\mathbb{Z}/_{n\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}/_{m\mathbb{Z}}$ is generated by elements of the form $a\otimes b$

$a\otimes b = ab\,(1\otimes 1)$. So it is a cyclic group, generated by $1\otimes 1$.

$$n(1\otimes 1) = (n1)\otimes 1 = 0 \quad \text{and} \quad m(1\otimes 1) = 1\otimes (m1) = 0$$

So $o(1\otimes 1) \mid \gcd(n,m)$; and there is an onto map

$$\mathbb{Z}/_{\gcd(n,m)}\mathbb{Z} \xrightarrow{g} \mathbb{Z}/_{n\mathbb{Z}} \otimes \mathbb{Z}/_{m\mathbb{Z}}$$
$$1 \longmapsto 1\otimes 1.$$

Let $f: \mathbb{Z}/_{n\mathbb{Z}} \times \mathbb{Z}/_{m\mathbb{Z}} \longrightarrow \mathbb{Z}/_{\gcd(n,m)}\mathbb{Z}$,

$$f(a+n\mathbb{Z}, b+m\mathbb{Z}) := ab + \gcd(n,m)\mathbb{Z}.$$

One can easily check that (1) f is well-defined.

(2) f is balanced

(3) f is bi-linear

$$\Rightarrow \exists\, \tilde{f}: \mathbb{Z}/_{n\mathbb{Z}} \otimes \mathbb{Z}/_{m\mathbb{Z}} \longrightarrow \mathbb{Z}/_{\gcd(n,m)}\mathbb{Z}, \quad (\text{homomo.})$$
$$\tilde{f}(\bar{a}\otimes\bar{b}) = \overline{ab}.$$

So $\tilde{f}$ is the inverse of g; this implies the claim.

Lecture 18: Examples of tensor product

Thursday, February 15, 2018 12:01 AM

Ex $\mathbb{Q}\otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$

Pf. $r\otimes x = \frac{r}{n}\otimes nx = 0$.

$\forall x\in \mathbb{Q}/\mathbb{Z}, \exists n\in\mathbb{Z}^+$ s.t. $nx=0$

Proposition. Suppose $I\lhd R$. Then $R/I\otimes_R M \simeq M/IM$ as R/I-mod.

$\bar{r}\otimes x \mapsto rx+IM$

Pf. Let $f: M\to R/I\otimes_R M$, $f(x):=\bar{1}\otimes x$. Then f is an R-mod. homomorphism:

$$f(r_1m_1+r_2m_2) = \bar{1}\otimes(r_1m_1+r_2m_2) = \bar{1}\otimes r_1m_1+\bar{1}\otimes r_2m_2$$
$$=\bar{r_1}\otimes m_1+\bar{r_2}\otimes m_2 = r_1(\bar{1}\otimes m_1)+r_2(\bar{1}\otimes m_2)$$
$$=r_1f(m_1)+r_2f(m_2).$$

$IM\subseteq \ker f$. Suppose $a\in I$, $m\in M$. Then

$$f(am) = \bar{1}\otimes am = \bar{a}\otimes m = 0.$$

So $\bar{f}: M/IM\to R/I\otimes_R M$, $\bar{f}(x+IM)=\bar{1}\otimes x$ is a well-def. R-mod. homomorphism.

Let $g: R/I\times M\to M/IM$, $g(r+I,m) := rm+IM$. One can check that g is a well-defined R-balanced, bilinear.

So $\exists \tilde{g}: R/I\otimes_R M\to M/IM$, $\tilde{g}(\bar{r}\otimes m) = rm+IM$.

Hence $\tilde{g}$ is the inverse of $\bar{f}$; and the claim follows. ∎

Lecture 18: Extension of scalars or base change

Tuesday, February 20, 2018 9:57 AM

Suppose $\phi: S \to R$ is a ring homomorphism (think about the case $S \hookrightarrow R$ where S is a subring of R), and M is a right S-mod. Then we can extend the scalar multiplicatoin from S to R or change the base ring from S to R:

we view R as an (S,R)-bi-module: $s \cdot r \cdot r' := \phi(s) r r'$

Now we can consider $M \otimes_S R$ which is a right R-module.

And $f: M \longrightarrow M \otimes_S R$, $f(x) := x \otimes 1$ satisfies

$$f(x_1 s_1 + x_2 s_2) = (x_1 s_1 + x_2 s_2) \otimes 1$$

$$= x_1 \otimes \phi(s_1) + x_2 \otimes \phi(s_2)$$

$$= f(x_1)\phi(s_1) + f(x_2)\phi(s_2).$$

As we have seen in the previous examples, f is not necessarily injective: $\mathbb{Q}/\mathbb{Z} \xrightarrow{f} \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the zero homomorphism.

Let M be an (S,R)-bimodule. Then for any right S-module N we got a right R-module $N \otimes_S M$. Do we get a functor from right S-Mod to right R-Mod?

Lecture 18: The tensor of two homomorphisms

Tuesday, February 20, 2018 10:12 AM

Lemma Suppose $f \in \mathrm{Hom}_S(N_1, N_2)$, and $g \in \mathrm{Hom}_{(S,R)}(M_1, M_2)$.

Then $\exists!\ f \otimes g \in \mathrm{Hom}_R(N_1 \otimes_S M_1, N_2 \otimes_S M_2)$ s.t.

$$(f \otimes g)(n_1 \otimes m_1) = f(n_1) \otimes g(m_1).$$

Pf. Let $\psi : N_1 \times M_1 \longrightarrow N_2 \otimes_S M_2$ be $\psi(n_1, m_1) := f(n_1) \otimes g(m_1)$.

Then $\psi(n_1 s, m_1) = f(n_1 s) \otimes g(m_1) = f(n_1) s \otimes g(m_1)$

$$= f(n_1) \otimes s\, g(m_1) = f(n_1) \otimes g(s m_1)$$

$$= \psi(n_1, s m_1)$$

$$\psi(n_1 - n_2, m) = f(n_1 - n_2) \otimes g(m) = (f(n_1) - f(n_2)) \otimes g(m)$$

$$= f(n_1) \otimes g(m) - f(n_2) \otimes g(m)$$

$$= \psi(n_1, m) - \psi(n_2, m)$$

$$\psi(n, m_1 r_1 + m_2 r_2) = f(n) \otimes g(m_1 r_1 + m_2 r_2)$$

$$= f(n) \otimes (g(m_1) r_1 + g(m_2) r_2)$$

$$= (f(n) \otimes g(m_1)) r_1 + (f(n) \otimes g(m_2)) r_2$$

$$= \psi(n, m_1) r_1 + \psi(n, m_2) r_2.$$

So $\exists!\ f \otimes g \in \mathrm{Hom}_R(N_1 \otimes_S M_1, N_2 \otimes_S M_2)$, $(f \otimes g)(n_1 \otimes m_1) = f(n_1) \otimes g(m_1)$. ∎

Lecture 18: Tensor functor

Tuesday, February 20, 2018 1:56 PM

Lemma. Let M be an (S,R)-bimodule. Then

$-\otimes_S M$: right S-mod $\longrightarrow$ right R-mod is a functor.

Pf. Suppose $N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3$ are right S-module hom.

Then
$$N_1 \otimes_S M \xrightarrow{f_1 \otimes id_M} N_2 \otimes_S M \xrightarrow{f_2 \otimes id_M} N_3 \otimes id_M$$
$$n_1 \otimes m \longmapsto f_1(n_1) \otimes m \longmapsto f_2(f_1(n_1)) \otimes m$$

So by the uniqueness of the previous lemma:

$$(f_2 \otimes id_M) \circ (f_1 \otimes id_M) = (f_2 \circ f_1) \otimes id_M ,$$

which shows $-\otimes_S M$ is a functor. ■

In the next lecture we show it is right exact.