Math200b, homework 5

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Tensor and direct sum.

You do not have to write anything for this part; only justify and understand all the statements. For two functors F_1 and F_2, we say $F_1 \simeq F_2$ when there is a natural isomorphism $\eta : F_1 \longrightarrow F_2$. Suppose $\{M_i\}_{i\in I}$ is a family of left A-modules and ${}_BN_A$ is a (B, A)-bimodule.

1. Suppose $\{F_i\}_{i\in I}$ is a family of functors from **left** A-**mod** to $\mathbf{Ab}$. Define the functor $\prod_{i\in I} F_i$.

2. Prove that

$$\prod_{i\in I} h^{M_i} \simeq h^{\bigoplus_{i\in I} M_i}.$$

3. Consider h^N : **left** B-**mod** $\to$ **left** A-**mod** ; and deduce

$$\begin{aligned} h^{\bigoplus_{i\in I} N\otimes_A M_i} \simeq \prod_{i\in I} h^{N\otimes_A M_i} &\simeq \prod_{i\in I} (h^{M_i} \circ h^N) \\ &\simeq (\prod_{i\in I} h^{M_i}) \circ h^N \simeq h^{\bigoplus_{i\in I} M_i} \circ h^N \\ &\simeq h^{N\otimes_A (\bigoplus_{i\in I} M_i)}. \end{aligned}$$

4. Prove that $\bigoplus_{i\in I}(N \otimes_A M_i) \simeq N \otimes_A (\bigoplus_{i\in I} M_i)$ as left B-modules. (During lecture we gave an alternative proof for the case of finite index set I.)

Localization and tensor product.

Suppose A is a unital commutative ring, $S \subseteq A$ is a multiplicatively closed subset, and M is an A-module. Convince yourself that localizing defines an exact functor from A-**mod** to $S^{-1}A$-**mod**.

1. Prove that $S^{-1}A \otimes_A M \simeq S^{-1}M$; deduce that $S^{-1}A$ is a flat A-module.

2. Prove that, if M is a flat A-module, then $S^{-1}M$ is a flat $S^{-1}A$-module.

3. Prove that $S^{-1}(M_1 \otimes_A M_2) \simeq S^{-1}M_1 \otimes_{S^{-1}A} S^{-1}M_2$, $\frac{x_1 \otimes x_2}{1} \mapsto \frac{x_1}{1} \otimes \frac{x_2}{1}$ as $S^{-1}A$-modules.

4. Prove that, if $M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module for any $\mathfrak{p} \in \mathrm{Spec}(A)$, then M is flat. (Hint: look at HW3, Localizing a module.)

More on flat modules.

1. Suppose k is a field and V and W are two k-vector spaces. Prove that $\dim_k(V \otimes_k W) = (\dim_k V)(\dim_k W)$.

2. Suppose A is a local unital commutative ring, M and N are finitely generated A-modules, and $M \otimes_A N = 0$. Prove that either $M = 0$ or $N = 0$. (Hint. Suppose $\mathrm{Max}(A) = \{\mathfrak{m}\}$. Let $k := A/\mathfrak{m}$. Argue $M/\mathfrak{m}M \simeq M \otimes_A k$ and $N/\mathfrak{m}N \simeq N \otimes_A k$. Show that $(M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N) =$

0.) (Notice that $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$ and so it is crucial that A is local. For an arbitrary ring A, we deduce that $M \otimes_A N = 0$ implies for any $\mathfrak{p} \in \operatorname{Spec} A$ either $M_{\mathfrak{p}} = 0$ or $N_{\mathfrak{p}} = 0$.)

3. Suppose A is a unital ring, N is a left A-modules, $0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$ is a S.E.S. of right A-modules, and M_3 is a flat A-module. Prove that

$$0 \longrightarrow M_1 \otimes_A N \xrightarrow{f_1 \otimes \mathrm{id}_N} M_2 \otimes_A N \xrightarrow{f_2 \otimes \mathrm{id}_N} M_3 \otimes_A N \longrightarrow 0$$

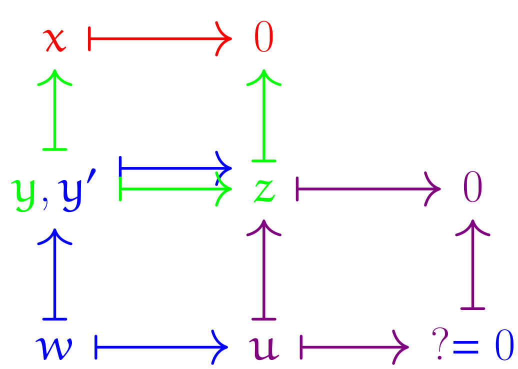
is a S.E.S.. (**Hint.** Argue that there is a S.E.S.

$$0 \longrightarrow K \xrightarrow{i} F \xrightarrow{\pi} N \longrightarrow 0$$

such that F is a free left A-module. Discuss why we get the following commuting diagram where all the rows and columns are exact. Suppose x is in the kernel of $f_1 \otimes \mathrm{id}_N$

and use the diagram to deduce $x = 0$.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M_1 \otimes_A N & \longrightarrow & M_2 \otimes_A N & \longrightarrow & M_3 \otimes_A N & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M_1 \otimes_A F & \longrightarrow & M_2 \otimes_A F & \longrightarrow & M_3 \otimes_A F & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M_1 \otimes_A K & \longrightarrow & M_2 \otimes_A K & \longrightarrow & M_3 \otimes_A K & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}
$$



Start with red, deduce existence of green, get the violet part, continue with blue. Argue why $y = y'$; and deduce that $x = 0$.

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(Notice that during lecture we proved

$$0 \to M_1 \otimes_A N \xrightarrow{j \otimes \mathrm{id}_N} (M_1 \oplus M_3) \otimes_A N \xrightarrow{p \otimes \mathrm{id}_N} M_3 \otimes_A N \to 0$$

is a S.E.S.; so we have already proved the above statement when M_3 is projective.)

4. Suppose A is a unital ring, $0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$ is a S.E.S. of right A-modules, and M_3 is a flat A-module. Prove that M_1 is flat if and only if M_2 is flat. (**Hint.** Use the previous problem and the Short Five Lemma.)