Lecture 04: Submodule generated by a subset

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<u>Def</u>. Let M be an A-module and $\emptyset \neq X \subseteq M$. Then the A-mod generated by X is the smallest submodule of M that contains X. And it is denoted by AX or $\langle X \rangle$.

We need to show that AX exist. We start with the following lemma:

<u>Lemma</u>. Suppose M is a left A-mod and $\{M_i\}_{i\in I}$ is a family of submodules of M. Then $\bigcap_{i\in I} M_i$ is a submodule.

<u>Pf</u>. • $x, y \in \bigcap_{i\in I} M_i \Rightarrow \forall i \in I,\ x, y \in M_i \Rightarrow \forall i \in I,\ x-y \in M_i$

$$\Rightarrow x-y \in \bigcap_{i\in I} M_i .$$

• $\begin{matrix} x \in \bigcap_{i\in I} M_i \\ a \in A \end{matrix} \Rightarrow \left.\begin{matrix} \forall i \in I,\ x \in M_i \\ a \in A \end{matrix}\right\} \Rightarrow \forall i \in I,\ ax \in M_i \Rightarrow ax \in \bigcap_{i\in I} M_i .$ ■

<u>Cor</u>. For any $\emptyset \neq X \subseteq M$, AX exists.

<u>Pf</u>. Let $N := \bigcap_{\substack{X \subseteq M' \\ M' \subseteq M \\ \text{submod.}}} M'$. Then $X \subseteq M'$ and by the previous lemma N is a submod. Moreover, if $X \subseteq M'$ and M' is a submod, then $N \subseteq M'$. Hence N is the smallest such submod. ■

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Next we describe elements of AX.

<u>Lemma</u>. Suppose M is a left A-mod and $\emptyset \neq X \subseteq M$. Then

$$AX = \{\sum_{i=1}^{n} a_i x_i \mid n \in \mathbb{Z}^+, a_i \in A, x_i \in X\}.$$

<u>Pf</u>. Let N be the RHS.

$\forall x_i \in X, a_i \in A$, $a_i x_i \in AX$ as AX is a left A-mod

$\Rightarrow \sum_{i=1}^{n} a_i x_i \in AX$ as AX is a subgroup

$\Rightarrow N \subseteq AX$. (I)

• One can easily see that N is a submodule and $\forall x \in X$, $x = 1.x \in N$; and so $X \subseteq N$ and N is a left A-submod. Therefore $AX \subseteq N$. (II)

By (I) and (II), $AX = N$. ∎

<u>Lemma</u>. Suppose M is a left A-mod and $\{M_i\}_{i\in I}$ is a family of submod of M. $A(\bigcup_{i\in I} M_i)$ is denoted by $\sum_{i\in I} M_i$. Then

$$\sum_{i\in I} M_i = \{\sum_{i\in I} m_i \mid \forall i \in I, m_i \in M_i \text{; except for finitely many } i \; \underset{m_i=0}{}\}.$$

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Pf. $\sum_{i\in I} M_i = A(\bigcup_{i\in I} M_i) = \{\sum_{j=1}^{n} a_j x_j \mid n\in\mathbb{Z}^+, a_j\in A, x_j\in\bigcup_{i\in I} M_i\}$

$x_j \in \bigcup_{i\in I} M_i \Rightarrow \exists\, i_j\in I,\ x_j\in M_{i_j} \Rightarrow \underbrace{a_j x_j}_{m_{i_j}} \in M_{i_j}$.

$$\Rightarrow \sum_{j=1}^{n} a_j x_j = \sum_{i\in I} m_i \text{ where}$$

$$m_{i_j} = a_j x_j \text{ and } m_i = 0 \text{ for } i\in I\setminus\{i_1,\dots,i_n\}$$

And so

$$A(\bigcup_{i\in I} M_i) \subseteq \{\sum_{i\in I} m_i \mid m_i\in M_i \text{ and only finitely many terms are non-zero}\}.$$

On the other hand $\sum_{i\in I} m_i = m_{i_1} + \cdots + m_{i_n} \in A(\bigcup_{i\in I} M_i)$

as $m_{i_j} \in \bigcup_{i\in I} M_i$ (let $a_j := 1$.); and claim follows ∎

Def. Suppose $\{M_i\}_{i\in I}$ is a family of submod. of M. We say

$\sum_{i\in I} M_i$ is the <u>internal direct sum</u> of M_i's and denote it

by $\bigoplus_{i\in I} M_i$ if

$$\sum_{i\in I} m_i = \sum_{i\in I} m_i' \Rightarrow m_i = m_i' \text{ for any } i\in I.$$

(except finitely many terms the rest are 0.)

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__Lemma__. Suppose $\{M_i\}_{i\in I}$ is a family of submod. of M. Then the following are equivalent:

(a) $\sum_{i\in I} M_i$ is an internal direct sum.

(b) $\forall j\in I,\ M_j \cap \sum_{i\in I\setminus\{j\}} M_i = 0$.

__Pf.__ (a) $\Rightarrow$ (b) $\quad m_j = \sum_{i\in I\setminus\{j\}} m_i \Rightarrow m_j = 0$ ✓

(b) $\Rightarrow$ (a)

$$\sum m_i = \sum m_i' \Rightarrow \forall j,\ m_j - m_j' \in M_j \cap \sum_{i\in I\setminus j} M_i = 0$$

$$\Rightarrow m_j - m_j' = 0 \Rightarrow m_j = m_j'.$$ ∎

Next we define __external direct sum__; that means we start with modules that are not necessarily submodules of an ambient module.

__Def.__ Suppose $\{M_i\}_{i\in I}$ is a family of left A-modules. Let

$\prod_{i\in I} M_i := \{(m_i)_{i\in I} \mid m_i \in I\}$ (it is called the direct product of M_i's.); let

$\bigoplus_{i\in I} M_i := \{(m_i)_{i\in I} \in \prod_{i\in I} M_i \mid$ except for finitely many i's the rest of components are $0\}$.

__Lemma__. $(m_i)_{i\in I} + (m_i')_{i\in I} := (m_i + m_i')_{i\in I}$ and $a.(m_i)_{i\in I} := (a m_i)_{i\in I}$

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make $\prod_{i\in I} M_i$ a left A-mod and $\bigoplus_{i\in I} M_i$ is a submod.

(It is clear!)

Theorem. (Universal Property of direct sum)

Suppose $\{M_i\}_{i\in I}$ is a family of left A-modules and M is a left A-module. Suppose, $\forall i\in I$, $\phi_i : M_i \to M$ is an A-mod. hom.

Then $\exists!\ \phi : \bigoplus_{i\in I} M_i \to M$ s.t. $\phi\circ j_i = \phi_i$ for any $i\in I$

where $j_i : M_i \to \bigoplus_{l\in I} M_l$, the i^{th} component of $j_i(x)$ is x

and other components are 0.

$$\begin{tikzcd} M_i \arrow[r, hook, "j_i"] \arrow[dr, "\phi_i"'] & \bigoplus_{l\in I} M_l \arrow[d, dashed, "\exists!\,\phi"] \arrow[dl, phantom, "\circlearrowleft"] \\ & M \end{tikzcd}$$

Pf. We start with uniqueness. If there is such ϕ, then

$$\phi((m_i)_{i\in I}) = \phi\Big(\underbrace{\sum_{i\in I} j_i(m_i)}_{\text{only finitely many terms are non-zero}}\Big) = \sum_{i\in I} (\phi\circ j_i)(m_i)$$

$$= \sum_{i\in I} \phi_i(m_i).$$ So ϕ is unique.

Existence. Let $\phi((m_i)_{i\in I}) := \sum_{i\in I} \phi_i(m_i)$. Since only finitely many m_i's are non-zero, this summation is legitimate. One can

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easily check that Φ satisfies the desired properties. ∎

<u>Remark 1</u>. Alternatively we can say

$$\begin{aligned} \mathrm{Hom}_A\Big(\bigoplus_{i\in I} M_i, M\Big) &\longrightarrow \prod_{i\in I} \mathrm{Hom}_A(M_i, M) \\ \Phi &\longmapsto (\Phi\circ j_i) \end{aligned}$$

is a bijection.

<u>Remark 2</u>. Free product of groups $\{G_i\}_{i\in I}$ plays the same role as external direct sum for groups; that means

$$\begin{aligned} \mathrm{Hom}\Big(\mathop{*}_{i\in I} G_i, H\Big) &\longrightarrow \prod_{i\in I} \mathrm{Hom}(G_i, H) \\ \Phi &\longmapsto (\Phi\circ j_i) \end{aligned}$$

is a bijection.

Next we justify why we use the same notation for internal and external direct sums.

<u>Proposition</u>. Suppose M is a left A-module and $\{M_i\}_{i\in I}$ is a family of submodules of M. Then

$\sum_{i\in I} M_i$ is an internal direct sum $\Leftrightarrow$

$f: \underset{\substack{(\text{external}\\ \text{direct sum})}}{\bigoplus_{i\in I} M_i} \to \sum_{i\in I} M_i$, $f((m_i)_{i\in I}) := \sum_{i\in I} m_i$ is an isomorphism.

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Pf. ($\Rightarrow$) • Since any element of $\sum_{i\in I} M_i$ is of the form $\sum_{i\in I} m_i$ for some $m_i \in M_i$ where m_i's are 0 except for finitely many i's, f is surjective.

• $f((m_i)_{i\in I}) = f((m_i')_{i\in I}) \Rightarrow \sum_{i\in I} m_i = \sum_{i\in I} m_i'$

$\underset{\text{(internal direct sum)}}{\Rightarrow}$ $\forall i\in I,\ m_i = m_i' \Rightarrow (m_i)_{i\in I} = (m_i')_{i\in I}$.

So f is injective.

• It is easy to check that f is an A-mod. hom.

($\Leftarrow$) $\sum_{i\in I} m_i = \sum_{i\in I} m_i'$ where m_i's and m_i''s are zero except for finitely many i's.

$\Rightarrow f((m_i)) = f((m_i')) \underset{\text{injective}}{\Longrightarrow} (m_i) = (m_i') \Rightarrow \forall i, m_i = m_i'$. ∎

Notice that there is a bijection between $\bigoplus_{i\in \mathbb{Z}^+} \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}^+$ ($(m_i)_{i\in\mathbb{Z}^+} \mapsto \prod p_i^{m_i}$ where p_i is the i^{th} prime.)

And there is a bijection between $\prod_{i\in\mathbb{Z}^+} \mathbb{Z}/2\mathbb{Z}$ and the power set $P(\mathbb{Z}^+)$ of $\mathbb{Z}^+$. So $\bigoplus_{i\in\mathbb{Z}^+} \mathbb{Z}/2\mathbb{Z}$ is countable and $\prod_{i\in\mathbb{Z}^+} \mathbb{Z}/2\mathbb{Z}$ is uncountable (by Cantor's theorem $|P(\mathbb{Z}^+)| > |\mathbb{Z}^+|$).

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Hence there is a big difference between $\bigoplus_{i\in I}$ and $\prod_{i\in I}$; of course when I is finite, they are the same.

<u>Def</u>. Suppose M is a left A-module and $\emptyset \neq X \subseteq M$. We say X freely generates a submodule if $AX = \sum_{x\in X} Ax$ is an internal direct sum, and $ax=0 \Rightarrow a=0$.

<u>Def</u>. For a non-empty set X and a unital ring A, the free A-module generated by X is denoted by $F(X)$ and

$$F(X) := \bigoplus_{x\in X} M_x \quad \text{where } M_x = Ax.$$

<u>Remark</u>. Here x is used more like a decoration!

So $A \xrightarrow{\sim} M_x$, $a \mapsto ax$. This is not needed; what we need is an embedding of X into $F(X)$. So we can assume $M_x := A$ and let $j: X \to \bigoplus_{x\in X} A$, s.t. the x^{th} component of $j(x)$ is 1 and other components of $j(x)$ are 0. In this form the above decoration x plays the role of $j(x)$.

Universal Property of free modules.

Suppose $\emptyset \neq X$, $j: X \to F(X)$, $x \mapsto$ in M_x-component and 0 other places (or $j(x)$ as described above). Suppose M is a left A-mod and $f: X \to M$ is a function. Then $\exists!\ \hat{f}: F(X) \to M$ s.t. $\hat{f}(j(x)) = f(x)$ for any $x \in X$.

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & F(X) \arrow[d, dashed, "\exists!\hat{f}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & M \end{tikzcd}$$

Set $\qquad$ A-mod

(We have seen existence of free groups; in the category of unital commutative rings, ring of poly in variables X is the free object.)

Pf. Again we start with uniqueness. If there is such an A-mod hom.,

$$\hat{f}((a_x)_{x\in X}) = \hat{f}\Big(\sum_{x\in X} a_x\, j(x)\Big)$$
$$= \sum_{x\in X} a_x\, \hat{f}(j(x))$$
$$= \sum_{x\in X} a_x f(x).$$

And so $\hat{f}$ is uniquely determined by f.

Existence. Let $\hat{f}((a_x)) := \sum_{x\in X} a_x f(x)$. It is easy to see

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that $\hat{f}$ satisfies the desired properties. ∎

Next we prove

<u>Theorem</u>. Suppose A is a unital commutative ring, $0\neq 1$. Then $A^n \simeq A^m$ if and only if $n=m$.

<u>Remark</u>. You will show in your HW that the above statement does not necessarily hold for non-commutative rings.

<u>Remark</u>. When $A=F$ is a field, using linear algebra we can define dimension of F^n and deduce the above result. This is used in the proof of the general case.

(We will prove this result in the next lecture.)