Math200b, lecture 9

Golsefidy

Exact sequences.

As in group theory, we use exact sequencers in order to split a problem on modules into easier pieces; and sometimes reduce it to a problem about simple modules.

Defintion. (a) We say $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n$ is an exact sequence if M_i's are (left) A-modules, $f_i \in \mathrm{Hom}_A(M_i, M_{i+1})$, and $\mathrm{Im} f_i = \ker f_{i+1}$; in particular $f_{i+1} \circ f_i = 0$.

(b) An exact sequence of the form $0 \to M_1 \to M_2 \to M_3 \to 0$ is called a Short Exact Sequence (S.E.S.).

(c) (ϕ_1, ϕ_2, ϕ_3) is called a S.E.S. homomorphism if $\phi_i \in \mathrm{Hom}_A(M_i, M_i')$, the following diagram is commuting, and each row is a S.E.S.:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \\ 0 & \longrightarrow & M_1' & \longrightarrow & M_2' & \longrightarrow & M_3' & \longrightarrow & 0 \end{array}$$

(ϕ_1, ϕ_2, ϕ_3) is called a S.E.S. isomorphism if it is a S.E.S. homomorphism and ϕ_i's are isomorphisms. (It is equivalent to a better definition: there exists a S.E.S. homomorphism (ψ_1, ψ_2, ψ_3) in the reverse direction such that together with ϕ_i's one gets a commuting diagram.)

Lemma 1 (a) $0 \to M_1 \xrightarrow{f} M_2$ *is an exact sequence if and only if* f *is injective.*
(b) $M_1 \xrightarrow{f} M_2 \to 0$ *is an exact sequence if and only if* f *is surjective.*
(c) *Suppose* $0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$ *is a S.E.S.; then it is isomorphic to*

$$0 \to \mathrm{Im} f_1 \hookrightarrow M_2 \xrightarrow{\pi} M_2/\mathrm{Im} f_1 \to 0,$$

where π *is the quotient map.*

Proof. (a) $0 \to M_1 \xrightarrow{f} M_2$ is an exact sequence $\Leftrightarrow 0 = \ker f \Leftrightarrow f$ is injective.

(b) $M_1 \xrightarrow{f} M_2 \to 0$ is an exact sequence $\Leftrightarrow \mathrm{Im} f = \ker 0 = M_2 \Leftrightarrow$ f is surjective.

(c) By the first isomorphism theorem,

$$\overline{f_2} : M_2/\ker f_2 \longrightarrow \mathrm{Im} f_2, \overline{f_2}(x + \ker f_2) := f_2(x)$$

is a well-defined isomorphism. Since

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

is a S.E.S., f_1 is injective and f_2 is surjective, and $\mathrm{Im} f_1 = \ker f_2$. So let $\theta' := \overline{f_2}^{-1} : M_3 \xrightarrow{\sim} M_2/\mathrm{Im} f_1$; and notice that

$$\theta'(f_2(x)) = x + \mathrm{Im} f_1.$$

Since f_1 is injective, $\theta : M_1 \xrightarrow{\sim} \mathrm{Im} f_1, \theta(x) := f_1(x)$ is an isomorphism. Overall we get that the following is a commuting diagram and claim follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \longrightarrow & 0 \\
 & & \downarrow \wr\, \theta & & \downarrow \mathrm{id}_{M_2} & & \downarrow \wr\, \theta' & & \\
0 & \longrightarrow & \mathrm{Im} f_1 & \hookrightarrow & M_2 & \xrightarrow{\pi} & M_2/\mathrm{Im} f_1 & \longrightarrow & 0
\end{array}$$

■

As we have pointed out earlier, when we are working with a S.E.S.

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

we often want to gain some information on M_2 assuming we have already some knowledge on M_1 and M_3. The same is true for a S.E.S. homomorphism (ϕ_1, ϕ_2, ϕ_3). The next lemma is a perfect example of such a result.

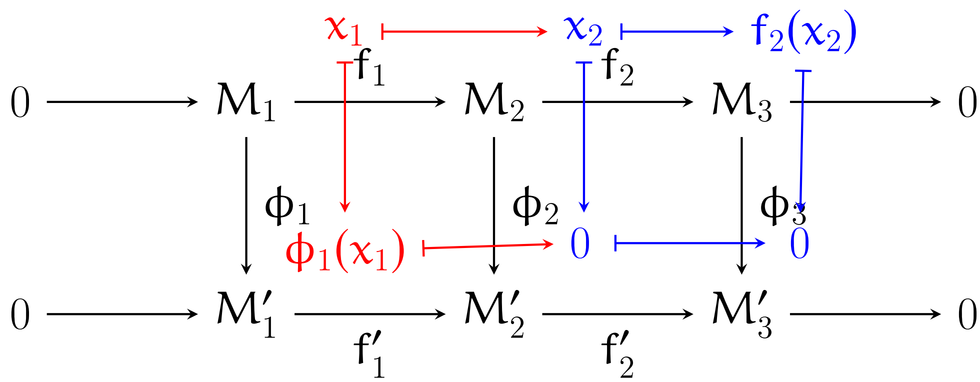
Lemma 2 (Short Five Lemma) *Suppose* (ϕ_1, ϕ_2, ϕ_3) *is a S.E.S. homomorphism. Then*

(a) *If* ϕ_1 *and* ϕ_3 *are injective, then* ϕ_2 *is injective.*

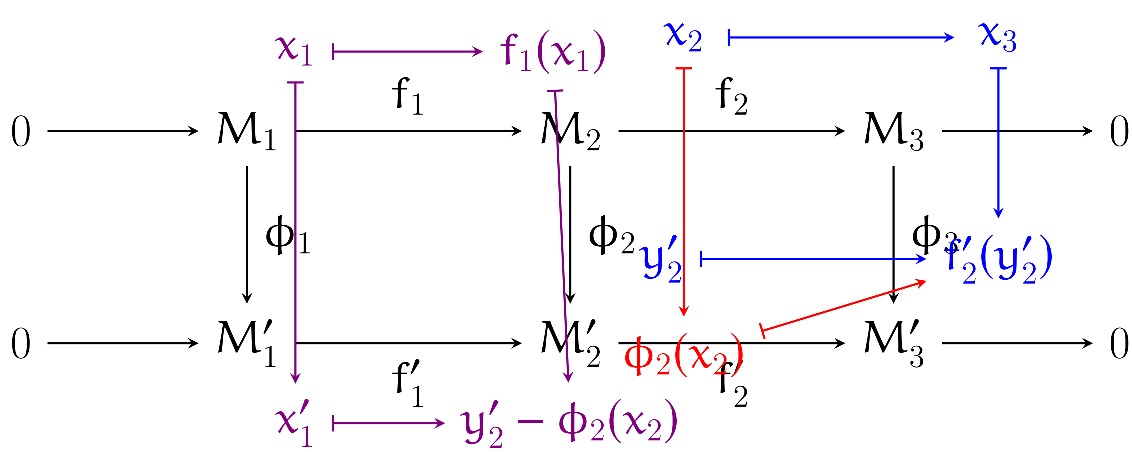
(b) *If* ϕ_1 *and* ϕ_3 *are surjective, then* ϕ_2 *is surjective.*

(c) *If* ϕ_1 *and* ϕ_3 *are isomorphisms, then* ϕ_2 *is an isomorphism.*

Proof. (a) Suppose $x_2 \in \ker \phi_2$. Then as you can see in the following diagram, $\phi_3(f_2(x_2)) = 0$. Since ϕ_3 is injective, $f_2(x_2) = 0$. Hence $x_2 \in \ker f_2 = \mathrm{Im} f_1$; say $x_2 = f_1(x_1)$. And so $f'_1(\phi_1(x_2)) = \phi_2(f_1(x_1)) = \phi_2(x_2) = 0$. Since f'_1 and ϕ_1 are injective, we deduce that $x_1 = 0$. Thus $x_2 = f_1(x_1) = 0$; and claim follows.



(b) Suppose $y_2' \in M_2'$. Since ϕ_3 is surjective, there is $x_3 \in M_3$ such that $\phi_3(x_3) = f_2'(y_2')$. As f_2 is surjective, there is $x_2 \in M_2$ such that $f_2(x_2) = x_3$. Therefore $f_2'(y_2') = f_2'(\phi_2(x_2))$, which implies that $y_2' - \phi_2(x_2) \in \ker f_2' = \mathrm{Im} f_1'$. Since ϕ_1 is surjective, there is $x_1 \in M_1$ such that $\phi_1(x_1) = x_1'$. And so $y_2' - \phi_2(x_2) = \phi_2(f_1(x_1))$, which implies $y_2' = \phi_2(x_2 + f_1(x_1)) \in \mathrm{Im}\phi_2$; and claim follows.



(c) This is an immediate consequence of parts (a) and (b). ■

Remark. There is a version of Short Five Lemma that involves exact sequences of the form $M_1 \to M_2 \to M_3 \to M_4 \to M_5$; and that is why it is called Short Five Lemma.

One way to construct a S.E.S. with a given M_1 and M_3 is by considering $M_2 := M_1 \oplus M_3$, $x_1 \mapsto (x_1, 0)$, and $(x_1, x_3) \mapsto x_3$. In the next theorem, we will see four statements that imply a given S.E.S. is of this form.

Theorem 3 (Splitting conditions) *Suppose*

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

is a S.E.S.; then the following statements are equivalent.

(a) $\exists N \subseteq M_2$ *which is a submodule and* $M_2 = N \oplus \mathrm{Im} f_1$.

(b) $\exists g_1 : M_2 \to M_1$ *such that* $g_1 \circ f_1 = \mathrm{id}_{M_1}$.

(c) $\exists \phi : M_2 \xrightarrow{\sim} M_1 \oplus M_3$ *such that* $(\mathrm{id}_{M_1}, \phi, \mathrm{id}_{M_3})$ *is an isomorphism of S.E.S..*

(b) $\exists g_2 : M_3 \to M_2$ *such that* $f_2 \circ g_2 = \mathrm{id}_{M_3}$.

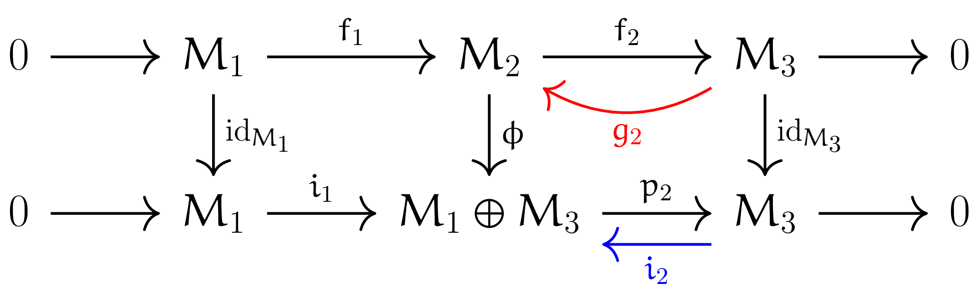
Proof. ((a) ⇒ (b)) Since f_1 is injective,

$$\overline{f_1} : M_1 \xrightarrow{\sim} \mathrm{Im} f_1, \overline{f_1}(x) := f_1(x)$$

is an isomorphism. Let $\pi : M_2 = N \oplus \mathrm{Im} f_1 \to \mathrm{Im} f_1$ be the projection to the second component; that means for $x \in N$ and $y \in \mathrm{Im} f_1$, we have $\pi(x + y) = y$. Let $g_1 := \overline{f_1}^{-1} \circ \pi : M_2 \to M_1$. It is easy to check that $g_1 \circ f_1 = \mathrm{id}_{M_1}$.

((b) $\Rightarrow$ (c)) Let $\phi : M_2 \to M_1 \oplus M_3$, $\phi(x_2) := (g_1(x_2), f_2(x_2))$. Then it is easy to see that $(\mathrm{id}_{M_1}, \phi, \mathrm{id}_{M_3})$ is a S.E.S. homomorphism. Since id_{M_1} and id_{M_3} are isomorphisms, by Short Five Lemma ϕ is an isomorphism; and claim follows.

((c) $\Rightarrow$ (d)) Following the arrows id_{M_3}, i_2, and ϕ^{-1}, we get the desired homomorphism g_2:



((d) $\Rightarrow$ (a)) Let $N := \mathrm{Im} g_2$. For any $x_2 \in M_2$, we have $f_2(g_2(f_2(x_2))) = f_2(x_2)$ as $f_2 \circ g_2 = \mathrm{id}_{M_3}$. Hence $x_2 - g_2(f_2(x_2)) \in \ker f_2 = \mathrm{Im} f_1$. Therefore $x_2 \in g_2(f_2(x_2)) + \mathrm{Im} f_1 \subseteq N + \mathrm{Im} f_1$, which implies

$$M_2 = \mathrm{Im} f_1 + N.$$

Suppose $x_2 \in \mathrm{Im} f_1 \cap N$. Hence $x_2 = g_2(x_3)$ for some $x_3 \in M_3$; and so $x_3 = f_2(g_2(x_3)) = f_2(x_2)$. Since $x_2 \in \mathrm{Im} f_1 = \ker f_2$, we have $f_2(x_2) = 0$. Overall we get $x_3 = f_2(x_2) = 0$. This implies that $x_2 = g_2(x_3) = g_2(0) = 0$; altogether $\mathrm{Im} f_1 \oplus N = M_2$. ■

Remark. For (non-commutative) groups, we called a S.E.S. split if (d) holds; and it only implies that the middle group is a semi-direct product of the other groups. If (a) holds, then the middle group is isomorphic to the direct product of the other groups; and its proof is similar to the argument presented above. It is worth pointing out that the above argument holds for groups as well; but it only implies that $\mathrm{Im} g_2$ is a subgroup of G_2 which is a *complement* of $\mathrm{Im} f_1$. Since $\mathrm{Im} g_2$ is not necessarily a normal subgroup, we can only deduce that their semi-direct product gives us G_2.

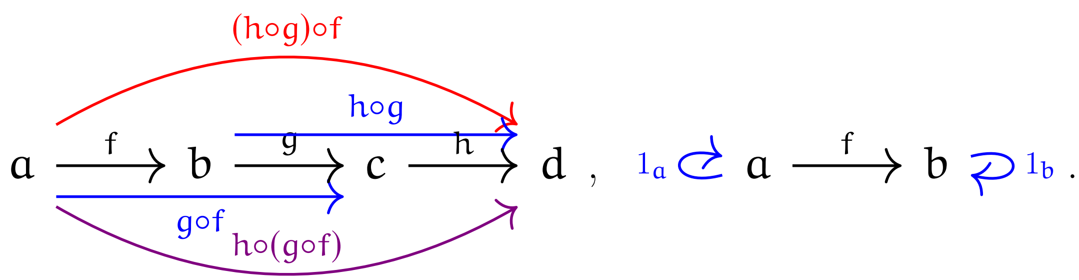
Basics of Category theory.

We only mention very basic concepts of Category Theory in this class and use it only as a language. A category $\mathcal{C}$ has a class of objects $\mathrm{Ob}(\mathcal{C})$ and for any two objects $a, b \in \mathrm{Ob}(\mathcal{C})$ it has a class of homomorphisms or arrows $\mathrm{Hom}_{\mathcal{C}}(a, b)$; with the following properties: for $a, b, c \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(a, b)$, and

$g \in \mathrm{Hom}_{\mathcal{C}}(b,c)$, there is $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(a,c)$ such that

(Associativity) $(f \circ g) \circ h = f \circ (g \circ h)$.

(Identity) For any $a \in \mathrm{Ob}(\mathcal{C})$, there is $1_a \in \mathrm{Hom}_{\mathcal{C}}(a,a)$ such that $1_a \circ f = f$ and $g \circ 1_a = g$ whenever they are defined. Alternatively the following diagrams are commuting.



Category $\mathcal{C}$ is called a small category if

$$\mathrm{Ob}(\mathcal{C}) \text{ and } \cup_{a,b \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(a,b)$$

are sets; it is called locally small if $\mathrm{Hom}_{\mathcal{C}}(a,b)$ is a set for any $a, b \in \mathrm{Ob}(\mathcal{C})$. In this course, we only work with locally small categories. Here are a few examples:

$\mathcal{Set}$. Objects are sets, and for any two sets a, b,

$$\mathrm{Hom}_{\mathcal{Set}}(a,b) := \{f : a \longrightarrow b \mid f \text{ is a function}\};$$

(with the caveat of the figuring out what it means to have a function to the empty set or from an empty set!)

$\mathbf{Grp}$. Objects are groups, and for any two groups a, b,

$$\mathrm{Hom}_{\mathbf{Grp}}(a, b) := \{f : a \to b | \ f \text{ is a group homomorphism}\}.$$

$\mathbf{Ab}$. Objects are abelian groups, and for any two abelian groups a, b,

$$\mathrm{Hom}_{\mathbf{Ab}}(a, b) := \{f : a \to b | \ f \text{ is a group homomorphism}\}.$$

$A-\mathbf{mod}$. Objects are groups, and for any two groups a, b,

$$\mathrm{Hom}_{A-\mathbf{mod}(a,b)} := \mathrm{Hom}_A(a, b).$$

One can think about a category as a directed graph with labeled edges (vertices are objects of the category, and edges are given by homomorphisms). Now having two such directed graphs, one can look for possible *graph homomorphisms*.

Suppose $\mathcal{C}$ and $\mathcal{D}$ are two categories. We say $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a functor if

(a) for any $a \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{F}(a) \in \mathrm{Ob}(\mathcal{D})$;

(b) for any $\phi \in \mathrm{Hom}_{\mathcal{C}}(a, b)$, $\mathcal{F}(\phi) \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(a), \mathcal{F}(b))$; and

(c) $\mathcal{F}(\phi \circ \psi) = \mathcal{F}(\phi) \circ \mathcal{F}(\psi)$ whenever they are defined.

(d) $\mathcal{F}(1_a) = 1_{\mathcal{F}(a)}$ for any $a \in \mathrm{Ob}(\mathcal{C})$

In the next lecture we will start with two general examples of functors.