Math200b, lecture 12

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Projective; but not free.

Example. Let $A = \mathbb{Z}[\sqrt{-10}]$ and $\mathfrak{a} := \langle 2, \sqrt{-10}\rangle$. Then $\mathfrak{a}$ is a projective A-module which is not free.

Proof. (Not free) This was proved in the previous lecture.

(Projective) It is enough to show $\mathfrak{a}$ is a direct summand of a free module. Notice that

$$0 \longrightarrow \ker\theta \hookrightarrow A^2 \xrightarrow{\theta} \mathfrak{a} \longrightarrow 0,$$

is a S.E.S., where $\theta(x_1, x_2) := 2x_1 + \sqrt{-10}x_2$. If we knew $\mathfrak{a}$ is projective, we could deduce that this sequence splits. On the other hand, if this sequence splits, then $\mathfrak{a}$ is a direct summand of the free module A^2; and so $\mathfrak{a}$ is projective. Hence it is necessary and sufficient to show that the above sequence splits. That means we need to find $\psi : \mathfrak{a} \longrightarrow A^2$ which is A-linear and $\theta(\psi(x)) = x$ for any $x \in \mathfrak{a}$. Thinking about $\mathfrak{a}$ as a subset of the field of fractions $F := \mathbb{Q}[\sqrt{-10}]$ of A and thinking about A^2 as a subset of $\mathbb{Q}[\sqrt{-10}]^2$, we see that A-linearity means $\psi(x) = (a_1 x, a_2 x)$ for some $a_1, a_2 \in \mathbb{Q}[\sqrt{-10}]$ (suppose $\psi_1 : \mathfrak{a} \longrightarrow A$ is the projection of ψ to the first component. Let $S := A \setminus \{0\}$. Then $S^{-1}\psi_1 : S^{-1}\mathfrak{a} \longrightarrow S^{-1}A$ is $S^{-1}A$-linear, which means $S^{-1}\psi : F \longrightarrow F$ is F-linear; hence $S^{-1}\psi_1(x) := a_1 x$ for some $a_1 \in F$.). So we are looking for $a_1, a_2 \in \mathbb{Q}[\sqrt{-10}]$ such that for any $x \in \mathfrak{a}$

$$a_1 x \in A, a_2 x \in A, \text{ and } (2a_1 + \sqrt{-10}a_2)x = x.$$

You can work out the details and find many such pairs. Here is one such example: $a_1 = 3, a_2 := \frac{\sqrt{-10}}{2}$; but let's explore these conditions a bit more. An alternative way of saying those conditions is

$$\exists\, a_1, a_2 \in \{a \in F |\ a\mathfrak{a} \subseteq A\}, \text{ and } 2a_1 + \sqrt{-10}a_2 = 1.$$

And this is equivalent to showing

$$\{a \in F |\ a\mathfrak{a} \subseteq A\}\mathfrak{a} = A.$$

In math200c, we will discuss fractional ideals (a A-submodule M of F such that $aM \subseteq A$ for some $a \in A$), define an equivalence relation on them ($M \sim N$ if $M = aN$ for some $a \in F^{\times}$), and get a semigroup structure. The above condition is the same as saying that $[\mathfrak{a}]$ is invertible. ■

Bimodules and representable functor.

As we have seen earlier, for an A-module M, the representable functor h^M indeed is a functor from A-**mod** to **Ab**; and I also pointed out that in the non-commutative setting $h^M(N) := \mathrm{Hom}_A(M, N)$ is not necessarily an A-module. Let's go over that argument again: for $\phi \in \mathrm{Hom}_A(M, N)$ and $a \in A$ one might want to define $(a \cdot \phi)(x) := a\phi(x)$. Then $(a \cdot \phi)(a'x) = a\phi(a'x) = aa'\phi(x)$ which is not necessarily $a'(a \cdot \phi)(x) = a'a\phi(x)$ (notice that if A is commutative, then it is fine and $\mathrm{Hom}_A(M, N)$ is an A-module). So we need "commuting actions".

Definition 1 *Suppose* A *and* B *are two unital rings. We say* M *is an* (A, B)*-bimodule if* M *is a left* A*-module and a right* B*-module,* *and for any* $a \in A$, $b \in B$, *and* $x \in M$, *we have* $(a \cdot x) \cdot b = a \cdot (x \cdot b)$.

Notice that if M is an (A, B)-bimodule, then $(a, b) \cdot x := a \cdot (x \cdot b)$ defines an $A \times B^{\mathrm{op}}$-module structure on M; and vice verse if M is an $A \times B^{\mathrm{op}}$-module, then it can be viewed as an (A, B)-bimodule.

Next we see that, if M is a (A,B)-bimodule, then $h^M(N)$ is a left B-module.

Proposition 2 *Suppose* M *is an* (A, B)*-bimodule; then*

$$h^M : \mathbf{left}\, A\text{-}\mathbf{mod} \to \mathbf{left}\, B\text{-}\mathbf{mod}.$$

is a functor.

Proof. We have already proved that $h^M : \mathbf{left}\, A\text{-}\mathbf{mod} \to \mathbf{Ab}$ is a functor. So to show the claim, it is enough to show $h^M(N)$ is a left B-module for any left A-module N, and, for any $\phi \in \mathrm{Hom}_A(M, N)$, $h^M(\phi)$ is a left B-module homomorphism.

In order to make $h^M(N) = \mathrm{Hom}_A(M, N)$ into a left B-module, we let $(b \cdot \phi)(m) := \phi(m \cdot b)$. Now we have to check that $b \cdot \phi$

is in $\mathrm{Hom}_A(M, N)$:

$$\begin{aligned}(b \cdot \phi)(m_1 + m_2) =& \phi((m_1 + m_2) \cdot b) = \phi(m_1 \cdot b + m_2 \cdot b) \\ =& \phi(m_1 \cdot b) + \phi(m_2 \cdot b) \\ =& (b \cdot \phi)(m_1) + (b \cdot \phi)(m_2).\end{aligned}$$

And

$$\begin{aligned}(b \cdot \phi)(a \cdot m) =& \phi((a \cdot m) \cdot b) && \text{(definition)} \\ =& \phi(a \cdot (m \cdot b)) && \text{(bimodule condition)} \\ =& a\phi(m \cdot b) && \text{(A-module homomorphism)} \\ =& a(b \cdot \phi)(m) && \text{(definition).}\end{aligned}$$

Next we check the B-module condition:

$$\begin{aligned}(b_1 \cdot (b_2 \cdot \phi))(m) =& (b_2 \cdot \phi)(m \cdot b_1) && \text{(definition)} \\ =& \phi((m \cdot b_1) \cdot b_2) && \text{(definition)} \\ =& \phi(m \cdot (b_1 b_2)) && \text{(B-mod property)} \\ =& ((b_1 b_2) \cdot \phi)(m) && \text{(definition).}\end{aligned}$$

The rest of the properties are similar if not easier. For ϕ in $\mathrm{Hom}_A(N, N')$, we have to show $h^M(\phi)$ is in $\mathrm{Hom}_B(N, N')$.

We have already proved that $h^M(\phi) \in \mathrm{Hom}_{\mathbb{Z}}(N, N')$. So it is enough to show $h^M(\phi)(b \cdot \psi) = b \cdot (h^M(\phi)(\psi))$.

$$
\begin{aligned}
(h^M(\phi)(b \cdot \psi))(m) &= \phi((b \cdot \psi)(m)) && \text{(def of functor)} \\
&= \phi(\psi(m \cdot b)) && \text{(def of module)} \\
&= (h^M(\phi)(\psi))(m \cdot b) && \text{(def of functor)} \\
&= (b \cdot (h^M(\phi)(\psi)))(m) && \text{(def of module)}.
\end{aligned}
$$

■

Since exactness of a chain of modules can be understood at the level of abelian groups, we deduce:

Corollary 3 *Suppose* M *is an* (A, B)*-bimodule; then*

$$h^M : \mathbf{left}\, A\text{-}\mathbf{mod} \to \mathbf{left}\, B\text{-}\mathbf{mod}.$$

is a left exact functor. And if M *is a projective left* A*-module, then* h^M *is an exact functor.*

Tensor product and Yoneda's lemma.

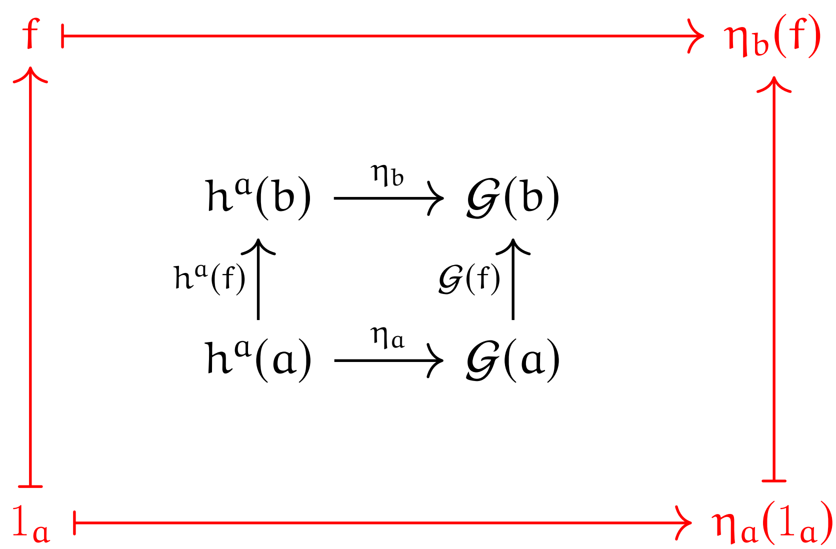
Suppose M is an (A, B)-bimodule and N is a left B-module. Then $h^N \circ h^M : \mathbf{left}\, A\text{-}\mathbf{mod} \to \mathbf{Ab}$ is a functor. Next we want to show that it is a representable functor. This means we have to show there is a left A-module $F(M,N)$ such that $h^{F(M,N)}(L)$ is naturally isomorphic to $(h^N \circ h^M)(L)$. So we need to find a natural transformation $\eta : h^{F(M,N)} \to h^N \circ h^M$ such that, for any L, $\eta_L : h^{F(M,N)}(L) \to h^N(h^M(L))$ is an isomorphism. To see how one can think about the set of natural transformations from a representable functor to another functor, we need to go over Yoneda's lemma.

Proposition 4 (Yoneda's lemma) *Suppose $\mathcal{C}$ is a locally small category. Then for any $a \in \mathrm{Ob}(\mathcal{C})$ and any functor $\mathcal{G} : \mathcal{C} \to \mathbf{Set}$, there is a bijection between the set $\mathrm{Nat}(h^a, \mathcal{G})$ of natural transformations from h^a to $\mathcal{G}$ and $\mathcal{G}(a)$.*

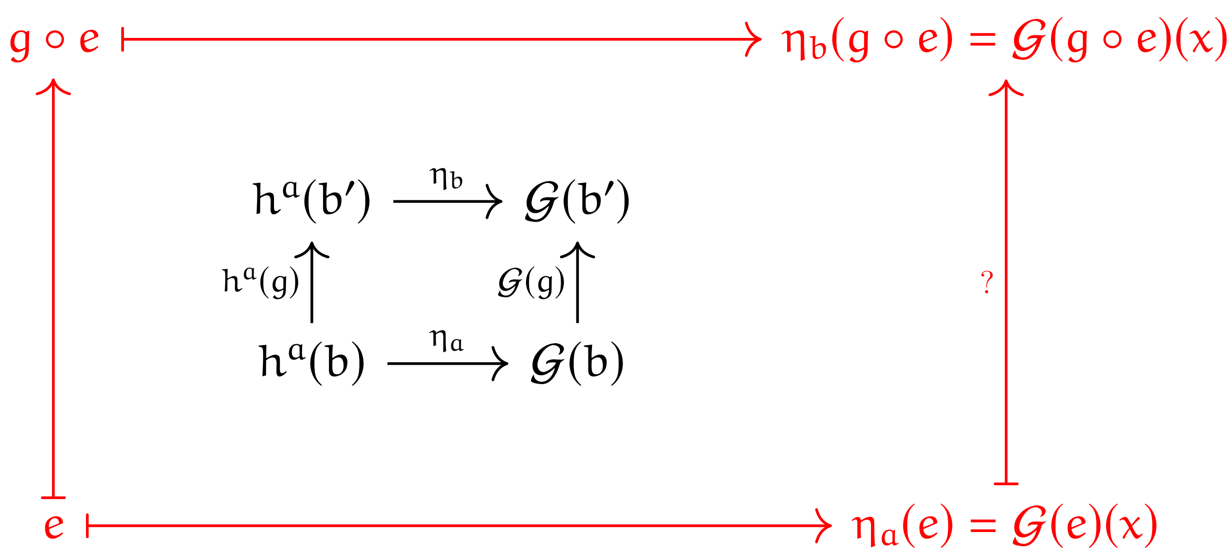
In fact this bijection is natural on a and $\mathcal{G}$.

Proof of Proposition 4. For $b \in \mathrm{Ob}(\mathcal{C})$ and $f \in h^a(b) = \mathrm{Hom}_{\mathcal{C}}(a, b)$, we need to find $\eta_b(f)$. As we can see below, $\eta_b(f) =$

$\mathcal{G}(f)(\eta_a(1_a))$; and so η is uniquely determined by $\eta_a(1_a) \in \mathcal{G}(a)$.



Conversely for $x \in \mathcal{G}(a)$, for any $f \in h^a(b)$, we can define $\eta_b(f) := \mathcal{G}(f)(x)$; and one can check that it defines a natural transformation: for $g \in \mathrm{Hom}_{\mathcal{C}}(b, b')$ we have to check:



This holds as $\mathcal{G}(g \circ f) = \mathcal{G}(g) \circ \mathcal{G}(f)$. ■

A general extremely vague phenomenon in mathematics is that *how an object interacts with itself determines how it interacts with others* (one's own worst enemy). You can see one instance of this phenomenon in Yoneda's lemma.

We will use the same idea as in proof of Yoneda's lemma to find $F(M, N)$ such that $h^{F(M,N)}$ becomes naturally isomorphic to $h^N \circ h^M$. We call $F(M, N)$ the tensor product of M and N over B, and it is denoted by $M \otimes_B N$. Along the way we show the universal property of tensor product, Tensor-Hom adjunction, and quickly deduce that the tensor product of two projective modules is projective (at least over a commutative ring).