Math200b, lecture 13

Golsefidy

Tensor product.

In the previous lecture we proved Yoneda's lemma which says there is a (natural) bijection between $\mathrm{Nat}(h^{a}, \mathcal{G})$ and $\mathcal{G}(a)$. Now we want to use the idea of Yoneda's proof to show for an (A, B)-bimodule M and a left B-module N, there is a left A-module $F(M, N)$ and a natural transformation

$$\eta : h^{F(M,N)} \to h^{N} \circ h^{M}$$

such that η_L is an isomorphism for any left A-module L. By Yoneda's lemma we know that η is uniquely determined by an element $f_0 \in h^N \circ h^M(F(M, N))$ using the following diagram:

$$\begin{array}{ccc} h^{F(M,N)}(L) & \xrightarrow{\eta_L} & h^N(h^M(L)) \\ {\scriptstyle h^{F(M,N)}(\phi)}\uparrow & & \uparrow{\scriptstyle h^N(h^M((\phi)))} \\ h^{F(M,N)}(F(M,N)) & \xrightarrow{\eta_{F(M,N)}} & h^N(h^M(F(M,N))) \end{array} \qquad \begin{array}{ccc} \phi & \longmapsto & \phi \circ f_0 \\ \uparrow & & \uparrow \\ 1_{F(M,N)} & \longmapsto & f_0. \end{array}$$

So we need to understand elements of $h^N(h^M(L))$; specially since we do not know what $F(M, N)$ is.

Suppose $\phi \in h^N(h^M(L)) = \mathrm{Hom}_B(N, \mathrm{Hom}_A(M, L))$; let

$$l_\phi : M \times N \to L, \, l_\phi(m, n) := (\phi(n))(m).$$

Then

(a) (Linear in N)

$$\begin{aligned} l_\phi(m, n_1 - n_2) =& (\phi(n_1 - n_2))(m) = (\phi(n_1) - \phi(n_2))(m) \\ =& (\phi(n_1))(m) - (\phi(n_2))(m) = \\ =& l_\phi(m, n_1) - l_\phi(m, n_2). \end{aligned}$$

(b) (A-Linear in M)

$$\begin{aligned} l_\phi(a_1 m_1 + a_2 m_2, n) =& (\phi(n))(a_1 m_1 + a_2 m_2) \\ =& a_1(\phi(n))(m_1) + a_2(\phi(n))(m_2) \\ =& a_1 l_\phi(m_1, n) + a_2 l_\phi(m_2, n). \end{aligned}$$

(c) (B-balanced)

$$\begin{aligned} l_\phi(m, b\cdot n) =& (\phi(b\cdot n))(m) = (b\cdot\phi(n))(m) \\ =& (\phi(n))(m\cdot b) = l_\phi(m\cdot b, n). \end{aligned}$$

One can easily see that the converse of this statement holds as well and we get

Proposition 1 *The following is a bijection from* $\mathrm{Hom}_B(N, \mathrm{Hom}_A(M, L))$ *and*

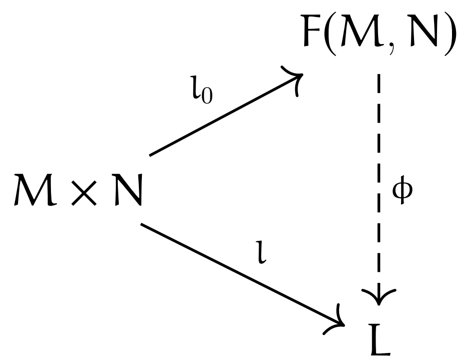
$\mathcal{B}_{M,N}(L) := \{l : M \times N \to L |$ *linear in* N, A-*linear in* M, B-*balanced*$\}$;

$\phi \mapsto l_\phi$ *where* $l_\phi(m, n) := (\phi(n))(m)$. *We denote its inverse by* $l \mapsto \phi_l$*; and so* $(\phi_l(n))(m) = l(m, n)$.

(Exercise: check the converse.)

So we need to find a left A-module $F(M, N)$ and $l_0 \in \mathcal{B}_{M,N}(F(M, N))$ such that for any $l \in \mathcal{B}_{M,N}(L)$ there is a unique $\phi \in \mathrm{Hom}_A(F(M, N), L)$ such that $l = \phi \circ l_0$: for l we get $\phi_l \in h^N(h^M(L))$, and so it is supposed to be $\phi \circ f_0$ for some unique $\phi \in \mathrm{Hom}_A(F(M, N), L)$; this means $\phi_l = \phi \circ f_0$ which implies that $l = \phi \circ l_0$.

So $(F(M,N), l_0)$ should have the following universal property: for any $l \in \mathcal{B}_{M,N}(L)$ there is a unique $\phi \in \mathrm{Hom}_A(F(M,N), L)$ such that the following diagram commutes:



Theorem 2 *For an* (A, B)*-bimodule* M *and a left* B*-module* N*, there is a unique* A*-module* $F(M, N)$ *and* $l_0 \in \mathcal{B}_{M,N}(F(M,N))$ *such that the above universal property holds.*

Proof. **(Existence)** Let $F(M \times N)$ be the free A-module generated by the set $M \times N$. Next we go to the largest quotient of $F(M \times N)$ such that $(m, n) \mapsto [(m, n)]$ becomes B-balanced, A-linear in M, and linear in N. So we let K be the A-submodule of $F(M \times N)$ that is generated by

$(m \cdot b, n) - (m, b \cdot n)$ (B-balanced)

$(a_1 m_1 + a_2 m_2, n) - a_1(m_1, n) - a_2(m_2, n)$ (A-linear in M)

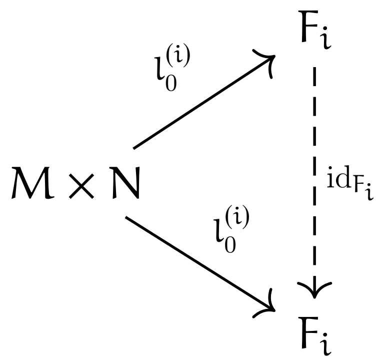
$(m, n_1 - n_2) - (m, n_1) + (m, n_2)$ (linear in N) for any $m, m_1.m_2 \in M$, $n, n_1, n_2 \in N$, $a_1, a_2 \in A$ and $b \in B$. And let $F(M,N) := F(M \times N)/K$, and

$$l_0 : M \times N \to F(M,N), l_0(m,n) := [(m,n)].$$

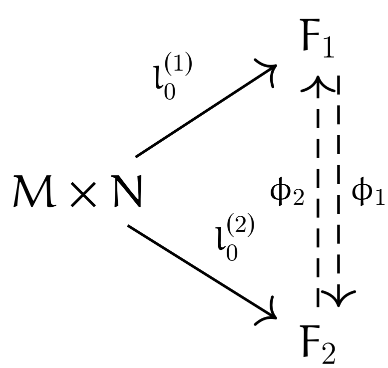
Then l_0 is in $\mathcal{B}_{M,N}(F(M,N))$. Suppose $l \in \mathcal{B}_{M,N}(L)$. By the universal property of free modules, there is an A-module homomorphism $\widehat{\phi} : F(M \times N) \to L$ such that $\widehat{\phi}(m,n) := l(m,n)$. Since $l \in \mathcal{B}_{M,N}(L)$, we can check that all the generators of K are in $\ker \widehat{\phi}$. Hence there is an A-module homomorphism $\phi : F(M,N) \to L$ such that $\phi([(m,n)]) = \widehat{\phi}(m,n) = l(m,n)$; and so $\phi \circ l_0 = l$. Since $F(M,N)$ is generated by the image of l_0, ϕ is uniquely determined by its values at $l_0(m,n)$'s; this implies the uniqueness of ϕ in the universal property.

(Uniqueness) Suppose $(F_1, l_0^{(1)})$ and $(F_2, l_0^{(2)})$ both satisfy the mentioned universal property. Because of the universal property, id_{F_i} is the unique A-module homomorphism from F_i to F_i

such that the following diagram commutes.



Since F_i's satisfy the universal property, there are A-module homomorphisms $\phi_1 : F_1 \longrightarrow F_2$ and $\phi_2 : F_2 \longrightarrow F_1$ such that the following diagram commutes



And so $\phi_1 \circ \phi_2$ and $\phi_2 \circ \phi_1$ are identities, which implies that they are isomorphisms. ■

The unique A-module $F(M, N)$ given in the above theorem is called the tensor product of M and N over B and it is denoted by $M \otimes_B N$. And $l_0(m, n)$ is denoted by $m \otimes n$ and it is called a pure tensor element.

To avoid confusion of all the involved left and right module structures, one can use the following notation: ${}_AM_B$ (for (A, B)-bimodule) and ${}_BN$ (for left B-module), now B's can help us glue these modules and end up getting a left A-module:

$$ {}_AM_B - -{}_BN \rightsquigarrow {}_AM \otimes_B N. $$

Similarly one can define for a right A-module P one can define

$$ P_A - -{}_AM_B \rightsquigarrow P \otimes_A M_B $$

which is a right B-module.

Let us summarize what we have proved:

Theorem 3 *Suppose ${}_AM_B$ is an (A, B)-bimodule and ${}_BN$ is a left B-module. Then there is a unique left A-module $M \otimes_B N$ that is generated by elements $\{m \otimes n\}_{m \in M, n \in N}$ such that*

(a) $(m, n) \mapsto m \otimes n$ *is a map from $M \times N$ to $M \otimes_B N$ that is B-balanced, A-linear in M, and linear in N.*

(b) **(Tensor-Hom adjunction)** *There is a natural isomorphism $\eta : h^{M \otimes_B N} \to h^N \circ h^M$; alternatively we say $\mathrm{Hom}_A(M \otimes_B N, L)$*

is naturally isomorphic to $\mathrm{Hom}_B(N, \mathrm{Hom}_A(M, L))$ *for any left* A*-module* L.

(c) **(Universal Property)** *For any* B*-balanced,* A*-linear in* M*, and linear in* N*, function* $l : M \times N \to L$ *there is a unique* $\phi : M \otimes_B N \to L$ *such that* $l(m, n) = \phi(m \otimes n)$.

Corollary 4 *Suppose* ${}_AM_B$ *is an* (A, B)*-bimodule and* ${}_BN$ *is a left* B*-module. If* M *is a projective* A*-module and* N *is a projective* B*-module, then* $M \otimes_B N$ *is a projective* A*-module.*

Proof. Since ${}_AM$ is projective and ${}_AM_B$ is a bimodule, h^M is an exact functor from **left** $\mathbf{A}$**-mod** to **left B-mod**. Since ${}_BN$ is a projective B-module, h^M is an exact functor from **left B-mod** to **Ab**. Hence $h^N \circ h^M$ is an exact functor from **left** $\mathbf{A}$**-mod** to **Ab**. ∎

The above corollary is particularly strong when A is a commutative ring. In this case, any module is both left and right A-module. Hence we can always talk about tensor product of two A-modules, and we get that tensor product of two projective A-modules is a projective A-module. So one can consider the set $K_0(A)$ (in what sense?) of finitely generated projective

A-modules up to isomorphism and define a semigroup structure on this set using tensor product. As you have seen in your HW assignment, any (f.g.) projective module is locally free. In math200c we focus on a subset of $K_0(A)$ that consists of (locally rank 1) invertible elements; this is called the Picard group $\mathrm{Pic}(A)$ of A.

In general it might be tricky to find various properties of a tensor product. Here is one example which shows how *torsion* elements might get killed in the tensor product.

Example. Show $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

Proof. Since $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ is generated by pure tensor elements, it is enough to show all pure tensor elements are zero. For any $r \in \mathbb{Q}$, $m, n \in \mathbb{Z} \setminus \{0\}$ we have

$$\begin{aligned} r \otimes \left(\frac{m}{n} + \mathbb{Z}\right) &= \left(\frac{r}{n}\right) n \otimes \left(\frac{m}{n} + \mathbb{Z}\right) && \mathbb{Z}\text{-balanced} \\ &= \left(\frac{r}{n}\right) \otimes n \left(\frac{m}{n} + \mathbb{Z}\right) \\ &= \left(\frac{r}{n}\right) \otimes 0. \end{aligned}$$

In any tensor product $a \otimes 0 = 0$; and this is because $a \otimes 0 + a \otimes 0 = a \otimes (0 + 0) = a \otimes 0$ (bilinear). And claim follows. ■