Math200b, lecture 14

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Tensor product: an example.

In the previous lecture we proved various properties of tensor product of two modules. We also mentioned that in general it is not that easy to describe various algebraic aspects of a tensor product; but certain examples play central role in this regard. Here is one of them:

Proposition 1 *Suppose* ${}_A M$ *is a left* A*-module and* $\mathfrak{a} \trianglelefteq A$*. Then*

$$\frac{A}{\mathfrak{a}} \otimes_A M \simeq \frac{M}{\mathfrak{a} M}$$

as A*-modules (or* $A/\mathfrak{a}$*-module).*

(Notice that $A/\mathfrak{a}$ can be considered an $(A/\mathfrak{a}, A)$-bimodule; and since $\mathfrak{a}(M/\mathfrak{a}M) = 0$, $M/\mathfrak{a}M$ can be considered a left $A/\mathfrak{a}$-module.)

Proof. Let $\widehat{\phi} : M \to A/\mathfrak{a} \otimes_A M, \widehat{\phi}(m) := 1 \otimes m$. Then

$$\begin{aligned}
\widehat{\phi}(a_1m_1 + a_2m_2) &= 1 \otimes (a_1m_1 + a_2m_2) && \text{(linear in } M\text{)} \\
&= (1 \otimes a_1m_1) + (1 \otimes a_2m_2) && \text{(}A\text{-balanced)} \\
&= ((a_1 + \mathfrak{a}) \otimes m_1) + ((a_2 + \mathfrak{a}) \otimes m_2) && \text{(}A\text{-linear)} \\
&= a_1(1 \otimes m_1) + a_2(1 \otimes m_2) \\
&= a_1\widehat{\phi}(m_1) + a_2\widehat{\phi}(m_2).
\end{aligned}$$

And so $\widehat{\phi}$ is an A-module homomorphism. Notice that for any $a \in \mathfrak{a}$ and $m \in M$ we have

$$\widehat{\phi}(am) = 1 \otimes am = (a + \mathfrak{a}) \otimes m = 0 \otimes m = 0.$$

Hence $\mathfrak{a}M \subseteq \ker \widehat{\phi}$; and so

$$\phi : M/\mathfrak{a}M \to A/\mathfrak{a} \otimes_A M, \phi(m + \mathfrak{a}M) := 1 \otimes m$$

is a well-defined (injective) A-module homomorphism. Next we use the universal property of tensor product to define an A-module homomorphism in the other direction.

Let $f : A/\mathfrak{a} \times M \to M/\mathfrak{a}M, f(a + \mathfrak{a}, m) := am + \mathfrak{a}M$.

Well-definedness. Suppose $a + \mathfrak{a} = a' + \mathfrak{a}$; then

$$a - a' \in \mathfrak{a} \Rightarrow (a - a')m \in \mathfrak{a}M \Rightarrow am - a'm \in \mathfrak{a}M.$$

It is even easier to check that f is A-balanced, A-linear in $A/\mathfrak{a}$, and linear in M. Hence by the universal property of tensor product, there is

$$\psi : A/\mathfrak{a} \otimes_A M \to M/\mathfrak{a}M, \psi((a+\mathfrak{a}) \otimes m) = f(a+\mathfrak{a}, m) = am+\mathfrak{a}M.$$

And so $\phi(\psi((a + \mathfrak{a}) \otimes m)) = (a + \mathfrak{a}) \otimes m$; since pure tensor elements generate the considered tensor product, $\phi \circ \psi$ is identity. We also have that $\phi \circ \psi$ is identity; hence ϕ and ψ are isomorphisms. ■

Example. Show that $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \simeq \mathbb{Z}/\gcd(m, n)\mathbb{Z}$ (as abelian groups).

Proof. By the previous proposition,

$$\begin{aligned}\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \simeq& \frac{\mathbb{Z}/m\mathbb{Z}}{n(\mathbb{Z}/m\mathbb{Z})} \\ =& \frac{\mathbb{Z}/m\mathbb{Z}}{(n\mathbb{Z} + m\mathbb{Z})/m\mathbb{Z}} \\ =& \frac{\mathbb{Z}/m\mathbb{Z}}{\gcd(m,n)\mathbb{Z}/m\mathbb{Z}} \\ \simeq& \frac{\mathbb{Z}}{\gcd(m,n)\mathbb{Z}}.\end{aligned}$$

■

Example. $f : M \to A \otimes_A M$, $f(m) := 1 \otimes m$ is an A-module isomorphism. (This is an immediate consequence of the above proposition; for $a = 0$.)

Base change.

Suppose $\phi : A \to B$ is a ring homomorphism; then B can be viewed as an (B, A)-bimodule: for $a \in A$, $b \in B$ and $x \in B$, let $x \cdot a := x\theta(a)$ and $b \cdot x := bx$. So for any left A-module M, we get a left B-module $B \otimes_A M$. We will see that it is in fact a functor from **left A-mod** to **left B-mod**. This is called a base change. Usually going the other direction is much harder;

starting with a B-module and trying to realize it as a base change of an A-module. This type of result is called descent. For instance when F is a subfield of E and $\theta : F \hookrightarrow E$ is the embedding F into E, this is part of Galois descent.

Tensor product as a functor.

Suppose ${}_AM_B$ is an (A, B)-bimodule; then for any left B-module N we get a left A-module $M \otimes_B N$. Can we make this into a functor from **left B-mod** to **left A-mod**? To get a functor, we have to say what it does to homomorphisms. We prove a stronger statement in this regard.

Proposition 2 *Suppose* $f \in \mathrm{Hom}_{(A,B)}(M, M')$ *and* $g \in \mathrm{Hom}_B(N, N')$*; then there is a unique element of* $\mathrm{Hom}_A(M \otimes_B N, M' \otimes_B N')$ *which sends* $m \otimes n$ *to* $f(m) \otimes g(n)$*. We denote this homomorphism by* $f \otimes g$.

Proof. We start with a B-balanced, A-linear in M, and linear in N, function from $M \times N$ to $M' \otimes_B N'$; and then use the universal property of tensor product to get the desired A-module homomorphism. Let

$$l : M \times N \to M' \otimes_B N', l(m,n) := f(m) \otimes g(n).$$

B-**balanced.**

$$\begin{aligned}
l(m \cdot b, n) =& f(m \cdot b) \otimes g(n) && \text{(right B-module hom)} \\
=& (f(m) \cdot b) \otimes g(n) && \text{(B-balanced)} \\
=& f(m) \otimes (b \cdot g(n)) && \text{(left B-module hom)} \\
=& f(m) \otimes g(b \cdot n) \\
=& l(m, b \cdot n).
\end{aligned}$$

A-**linear in** M.

$$\begin{aligned}
l(a_1 m_1 + a_2 m_2, n) =& f(a_1 m_1 + a_2 m_2) \otimes g(n) \\
=& (a_1 f(m_1) + a_2 f(m_2)) \otimes g(n) \\
=& a_1(f(m_1) \otimes g(n)) + a_2(f(m_2) \otimes g(n)) \\
=& a_1 l(m_1, n) + a_2 l(m_2, n).
\end{aligned}$$

Linear in N.

$$\begin{aligned} l(m, n_1 + n_2) =& f(m) \otimes g(n_1 + n_2) \\ =& f(m) \otimes (g(n_1) + g(n_2)) \\ =& f(m) \otimes g(n_1) + f(m) \otimes g(n_2) \\ =& l(m, n_1) + l(m, n_2). \end{aligned}$$

Hence by the universal property of tensor product there is a unique A-module homomorphism $\widehat{l} : M \otimes_B N \to M' \otimes_B N'$ such that $\widehat{l}(m \otimes n) = l(m, n) = f(m) \otimes g(n)$. ■

Theorem 3 *Suppose* ${}_A M_B$ *is an* (A, B)*-bimodule; then*

$$T_M : \textbf{left } B\text{-} \textbf{ mod} \to \textbf{left } A\text{-} \textbf{ mod}$$

is a functor where for any left B*-module* N, $T_M(N) := M \otimes_B N$ *and for any* $f \in \mathrm{Hom}_B(N, N')$, $T_M(f) := \mathrm{id}_M \otimes f$.

Proof. We have already showed that $T_M(N)$ is a left A-module, and $T_M(f) \in \mathrm{Hom}_A(M \otimes_B N, M \otimes_B N')$. So it is enough to show $T_M(f_1 \circ f_2) = T_M(f_1) \circ T_M(f_2)$ and $T_M(\mathrm{id}_N) = \mathrm{id}_{T_M(N)}$. Since pure tensor elements generate tensor product and $T_M(f_1 \circ f_2)$, $T_M(f_1) \circ T_M(f_2)$, and $T_M(\mathrm{id}_N)$ are A-module homomorphisms, it is enough to check the claim equalities for pure tensor elements.

$$\begin{aligned}(T_M(f_1) \circ T_M(f_2))(m \otimes n) =& T_M(f_1)((\mathrm{id}_M \otimes f_2)(m \otimes n)) \\ =& (\mathrm{id}_M \otimes f_1)(m \otimes f_2(n)) \\ =& m \otimes f_1(f_2(n)) \\ =& T_M(f_1 \circ f_2)(m \otimes n).\end{aligned}$$

And $T_M(\mathrm{id}_N)(m \otimes n) = (\mathrm{id}_M \otimes \mathrm{id}_N)(m \otimes n) = m \otimes n.$ ■

Tensor functor is right exact.

We have seen that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$; this shows that $T_{\mathbb{Q}/\mathbb{Z}}(j) = 0$ where $j : \mathbb{Z} \hookrightarrow \mathbb{Q}$. Notice that $T_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Z}) = \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \simeq \mathbb{Q}/\mathbb{Z}$; and so $T_{\mathbb{Q}/\mathbb{Z}}(j)$ is not injective though j is injective. So T_M is not necessarily left exact.

Theorem 4 (Tensor defines a right exact functor) *Suppose* ${}_A M_B$ *is an* (A, B)*-bimodule; then*

$$T_{{}_A M_B} : \textbf{left } B\text{-}\textbf{ mod} \to \textbf{left } A\text{-}\textbf{ mod}$$

is a right exact functor. (We often write T_M *instead of* $T_{{}_A M_B}$*.)*

Proof. Suppose $0 \longrightarrow N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3 \longrightarrow 0$ is a S.E.S. of left B-modules. Then $0 \longrightarrow T_M(N_1) \xrightarrow{T_M(f_1)} T_M(N_2) \xrightarrow{T_M(f_2)} T_M(N_3) \longrightarrow 0$ is a sequence of A-modules and A-module homomorphisms. Since $T_M(f_2) \circ T_M(f_1) = T_M(f_2 \circ f_1) = 0$, (it is a chain of A-modules and) $\mathrm{Im}(T_M(f_1)) \subseteq \ker T_M(f_2)$. So there is an A-module homomorphism

$$\theta : T_M(N_2)/\mathrm{Im}(T_M(f_1)) \longrightarrow T_M(N_3), \theta([x]) = T_M(f_2)(x),$$

where $[x] := x + \mathrm{Im}(T_M(f_1))$; in particular $\theta([m \otimes n_2]) = m \otimes f_2(n_2)$ where $[x] := x + \mathrm{Im}(T_M(f_1))$.

It is enough to show θ is an isomorphism.

By showing θ is an isomorphism, we deduce that θ is injective; and so $\ker T_M(f_2) = \mathrm{Im}(T_M(f_1))$. And surjectivity of θ implies that $T_M(f_2)$ is surjective.

To show θ is an isomorphism we will show that it has an inverse. We start by defining a suitable function from $M \times N_3$ to $T_M(N_2)/\mathrm{Im}(T_M(f_1))$; and then we use the universal property of tensor product in order to find the inverse of θ.

Let $l : M \times N_3 \longrightarrow T_M(N_2)/\mathrm{Im}(T_M(f_1)), l(m, n_3) := [m \otimes n_2]$ where $n_2 \in f_2^{-1}(n_3)$.

Well-definedness. Suppose $f_2(n_2) = f_2(n_2')$; then $n_2 - n_2' \in \ker f_2 = \mathrm{Im}(f_1)$. Hence $m \otimes n_2 - m \otimes n_2' \in \mathrm{Im}(T_M(f_1))$; and so $[m \otimes n_2] = [m \otimes n_2']$, which implies that l is well-defined.

B-**balanced.**

$$\begin{aligned} l(m \cdot b, n_3) =& [(m \cdot b) \otimes n_2] \\ =& [m \otimes b \cdot n_2] \quad \text{(since } f_2(b \cdot n_2) = b \cdot f_2(n_2) = b \cdot n_3\text{)} \\ =& l(m, n \cdot n_3). \end{aligned}$$

A-**linear in** M and **linear in** N_3 are clear.

Hence by the universal property of tensor product, there is an A-module homomorphism

$$\psi : M \otimes_B N_3 \longrightarrow T_M(N_2)/\mathrm{Im}(T_M(f_1)), \psi(m \otimes n_3) = [m \otimes n_2],$$

where $f_2(n_2) = n_3$. Notice that

$$\theta \circ \psi(m \otimes n_3) = \theta([m \otimes n_2]) = m \otimes f_2(n_2) = m \otimes n_3,$$

for any $m \in M$ and n_3. As pure tensor elements generate the tensor product as an A-module, we deduce that $\theta \circ \psi$ is identity. We also have

$$\psi \circ \theta([m \otimes n_2]) = \psi(m \otimes f_2(n_2)) = [m \otimes n_2];$$

and so $\psi \circ \theta$ is also identity. Therefore θ is an isomorphism. ■

Corollary 5 (Flat modules) *Suppose ${}_A M_B$ is an (A, B)-bimodule. Then the functor T_M is an exact functor if and only if $T_M(f)$ is injective for any injective homomorphism f. In this case, we say M is a flat B-module.*

Remark. As you can see, in the above definition, we say M is a flat B-module and there is no mention of A. This might need a justification that you will see in your HW assignment. Here is the statement that you will prove: there is a natural isomorphism between the functors

$$\begin{tikzcd} \textbf{left } B\textbf{-mod} \arrow[r, "T_{{}_A M_B}"] \arrow[rr, bend left, "F \circ T_{{}_A M_B}"] & \textbf{left } A\textbf{-mod} \arrow[r, "F"] & \mathbf{Ab} \end{tikzcd}$$

and

$$\textbf{left } B\textbf{-mod} \xrightarrow{T_{{}_{\mathbb{Z}} M_B}} \mathbf{Ab}$$

where F is the forgetful functor. Hence $F \circ T_{{}_A M_B}$ is exact if and only if $T_{{}_{\mathbb{Z}} M_B}$ is exact. On the other hand, exactness of a sequence of modules is determined at the level of abelian groups; hence $F \circ T_{{}_A M_B}$ is exact if and only if $T_{{}_A M_B}$ is exact. So overall we get

$$T_{{}_A M_B} \text{ is exact} \Leftrightarrow T_{{}_{\mathbb{Z}} M_B} \text{ is exact.}$$

And so flatness of M just depends on its B-module structure and is independent of its A-module structure.