I expect you to read the rest of Chapter 10.4. Proofs are just small variants of the arguments discussed in class.

1. Let $A \in M_n(\mathbb{C})$ and $R = \mathbb{C}[A]$ be the $\mathbb{C}$-subalgebra generated by A. Prove that A is diagonalizable if and only if R does not a nilpotent element.

2. In this problem, we would like to find an algorithm to compute the rational canonical form of a matrix:

Let F be a field and $A \in M_n(F)$. Then $V = F^n$ can be considered as an $F[x]$-mod via

$$x \cdot \vec{v} = A\vec{v},$$

and clearly V is generated by $e_1, \dots, e_n$, where

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \longrightarrow i^{th}.$$

Let $\phi : F[x]^n \longrightarrow F^n$ be the $F[x]$-mod. homomorphism such that $\phi(e_i) = e_i$ for $1 \leq i \leq n$.

ⓐ Prove that $\ker(\phi)$ is a free $F[x]$-mod and

$$\operatorname{rank}_{F[x]}(\ker(\phi)) = n.$$

ⓑ Prove that for any $\vec{w}(x) \in F[x]^n$ we have

$$\phi(A\,\vec{w}(x)) = A\,\phi(\vec{w}(x)).$$

(Hint: First observe that $\phi(\vec{w}_0) = \vec{w}_0$ if $\vec{w}_0 \in F^n$.
Then write $\vec{w}(x) = \sum_{i=0}^{m} x^i \vec{w}_i$ where $w_i \in F^n$.)

ⓒ Prove that $\ker(\phi) = \operatorname{Im}(xI - A)$, where

$$\operatorname{Im}(xI - A) = \{(xI - A)\vec{w} \mid \vec{w} \in F[x]^n\}.$$

(Here we are considering $xI - A$ as an element of $M_n(F[x])$.)

(Hint: Use part ⓑ and Problem 3 in the midterm exam.)

ⓓ Prove that if for some $P_1, P_2 \in GL_n(F[x])$

and $q_1(x) \mid q_2(x) \mid \cdots \mid q_n(x)$ we have

$$P_1(xI-A)P_2 = \begin{bmatrix} q_1(x) & & \\ & \ddots & \\ & & q_n(x) \end{bmatrix},$$

then $\{q_i(x) \mid q_i(x) \notin F\}$ is the set of invariant factors of A.

Remark. For a given matrix $A \in M_n(F)$, one can apply the elementary row and column operations on $xI-A$ to get a diagonal matrix. Part (d) says that the non-constant diagonal entries are the invariant factors of A.

3. Let $\{M_i\}_{i\in I}$ be a family of R-mod. Assume R is a subring of S and $1_R = 1_S$. Prove that

$$S \otimes_R \left(\bigoplus_{i\in I} M_i \right) \simeq \bigoplus_{i\in I} (S \otimes_R M_i).$$

4. Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}$.

5. Let $\mathbb{H} := \left\{ \begin{bmatrix} z & \omega \\ -\bar{\omega} & \bar{z} \end{bmatrix} \,\middle|\, z, \omega \in \mathbb{C} \right\}$.

(a) Prove that $\mathbb{H}$ is a division algebra.

(b) Prove that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$.

6. (Bonus Problem) Let $J_n(\lambda)$ be the Jordan block of size n with eigenvalue λ. Find the Jordan blocks of

$$J_n(0) \otimes J_m(0)$$

where $J_n(0) \otimes J_m(0) : \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^m \longrightarrow \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^m$ s.t.

$$\left(J_n(0) \otimes J_m(0)\right)(v \otimes w) := J_n(0)\, v \otimes J_m(0)\, w.$$

7. Let $j : 2\mathbb{Z} \longrightarrow \mathbb{Z}$ be the inclusion map. Is the $\mathbb{Z}$-module homomorphism

$$j \otimes \mathrm{id} : 2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

injective? Justify your answer.