1 Homework 4.

Reading on localization of rings and modules. We have seen how working with the field of fractions of an integral domain can help us study integral domains and their modules. Here, we extend that to other unital commutative rings. Suppose A is a unital commutative ring, $S \subseteq A$ is multiplicatively closed, and M is an A-module. We define *fractions* with numerators in either A or M and denominators in S by defining the following relations on $A \times S$ or $M \times S$, respectively. For $(a_1, s_1), (a_2, s_2) \in A \times S$, we say

$$(a_1, s_1) \sim (a_2, s_2) \Leftrightarrow \exists s \in S, s(s_2 a_1 - s_1 a_2) = 0,$$

and similarly for $(m_1, s_1), (m_2, s_2) \in M \times S$, we say

$$(m_1, s_1) \sim (m_2, s_2) \Leftrightarrow \exists s \in S, s(s_2 m_1 - s_1 m_2) = 0.$$

Check that $\sim$ is an equivalence relation. For all $a \in A$, $m \in$M, and $s \in S$, we let

$$\frac{a}{s} := [(a, s)]_\sim \quad \text{and} \quad \frac{m}{s} := [(m, s)]_\sim.$$

We let

$$S^{-1}A := \left\{ \frac{a}{s} \mid a \in A, s \in S \right\} \quad \text{and} \quad S^{-1}M := \left\{ \frac{m}{s} \mid m \in M, s \in S \right\}.$$

For all $a, a_1, a_2 \in A$, $m, m_1, m_2 \in M$, and $s, s', s_1, s_2 \in S$, we let

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} := \frac{s_2 a_1 + s_1 a_2}{s_1 s_2}, \quad \text{and} \quad \frac{a_1}{s_1}\frac{a_2}{s_2} := \frac{a_1 a_2}{s_1 s_2},$$

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} := \frac{s_2 m_1 + s_1 m_2}{s_1 s_2}, \quad \text{and} \quad \frac{a}{s} \cdot \frac{m}{s'} := \frac{a \cdot m}{s s'}.$$

Check that these are well-defined operations, $S^{-1}A$ is a ring, and $S^{-1}M$ is an $S^{-1}A$-module. The ring $S^{-1}A$ is called the *localization* of A by S, and $S^{-1}M$ is called the localization of M by S.

Suppose $\phi : M \to M'$ is an A-module homomorphism. Let

$$S^{-1}\phi\left(\frac{m}{s}\right) := \frac{\phi(m)}{s}.$$

Check that $S^{-1}\phi$ is a well-defined $S^{-1}A$-module homomorphism.

If N is a submodule of M, then $S^{-1}N$ is a submodule of $S^{-1}M$. Check that

$$\frac{S^{-1}M}{S^{-1}N} \to S^{-1}\left(\frac{M}{N}\right), \quad \frac{m}{s} + S^{-1}N \mapsto \frac{m+N}{s}$$

is a well-defined $S^{-1}A$-module isomorphism.

Notice that by the above statements for every ideal $\mathfrak{a}$ of A, $S^{-1}\mathfrak{a}$ is an ideal of $S^{-1}A$. Notice that if $\mathfrak{a} \cap S = \varnothing$, then

$$\overline{S} := \{s + \mathfrak{a} \mid s \in S\}$$

is a multiplictively closed subset of $A/\mathfrak{a}$. Check that

$$\frac{S^{-1}A}{S^{-1}\mathfrak{a}} \to \overline{S}^{-1}\left(\frac{A}{\mathfrak{a}}\right), \quad \frac{x}{s} + S^{-1}\mathfrak{a} \mapsto \frac{a+\mathfrak{a}}{s+\mathfrak{a}}$$

is a well-defined ring isomorphism.

Recall that if $\mathfrak{p}$ is a prime ideal, then $S_{\mathfrak{p}} := A \setminus \mathfrak{p}$ is a multiplicatively closed subset. The localization of A by $S_{\mathfrak{p}}$ is also denoted by $A_{\mathfrak{p}}$ and it is also called localization of A at $\mathfrak{p}$. Similarly, we write $M_{\mathfrak{p}} := S_{\mathfrak{p}}^{-1}M$ and call it the localization of M at $\mathfrak{p}$. If $\phi : M \to M'$ is an A-module homomorphism, then $S_{\mathfrak{p}}^{-1}\phi$ is denoted by $\phi_{\mathfrak{p}}$. So $\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ is an $A_{\mathfrak{p}}$-module homomorphism.

1. Suppose M is an A-module. Prove that the following statements are equivalent.

   (a) $M = 0$.

   (b) For all $\mathfrak{p} \in \operatorname{Spec}(A)$, $M_{\mathfrak{p}} = 0$.

   (c) For all $\mathfrak{m} \in \operatorname{Max}(A)$, $M_{\mathfrak{m}} = 0$.

   (**Hint**. For $x \in M$, consider $\operatorname{ann}(x) := \{a \in A \mid ax = 0\}$. Notice that $\operatorname{ann}(x)$ is an ideal of A. If $x \neq 0$, then there exists a maximal ideal $\mathfrak{m}$ such that $\operatorname{ann}(x) \subseteq \mathfrak{m}$. Get a contradiction using the assumption that $\frac{x}{1} \in M_{\mathfrak{m}}$ is 0.)

2. Suppose $\phi : M \to M'$ is an A-module homomorphism. Prove that ϕ is injective if and only if $\phi_{\mathfrak{m}}$ is injective for all $\mathfrak{m} \in \operatorname{Max}(A)$. (**Hint**. Show that $\ker(\phi_{\mathfrak{m}}) = (\ker \phi)_{\mathfrak{m}}$ and use the previous problem.)

3. Suppose $\phi : M \to M'$ is an A-module homomorphism. Prove that ϕ is surjective if and only if $\phi_{\mathfrak{m}}$ is surjective for all $\mathfrak{m} \in \mathrm{Max}(A)$. (**Hint**. Consider the co-kernel $M'/\mathrm{Im}(\phi)$ of ϕ. Notice that $(M'/\mathrm{Im}(\phi))_{\mathfrak{m}} \simeq M'_{\mathfrak{m}}/\mathrm{Im}(\phi)_{\mathfrak{m}}$.)

4. Suppose $\widetilde{\mathfrak{a}}$ is a proper ideal of $S^{-1}A$. Let

$$\mathfrak{a} := \{x \in A \mid \frac{x}{1} \in \widetilde{\mathfrak{a}}\}.$$

Prove that $\mathfrak{a}$ is an ideal of A, $\mathfrak{a} \cap S = \varnothing$, and $\widetilde{\mathfrak{a}} = S^{-1}\mathfrak{a}$.

5. Let $\mathscr{O}_S := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \cap S = \varnothing\}$. Let

$$\Theta : \mathscr{O}_S \to \mathrm{Spec}(S^{-1}A), \quad \Theta(\mathfrak{p}) := S^{-1}\mathfrak{p},$$

and

$$\Psi : \mathrm{Spec}(S^{-1}A) \to \mathscr{O}_S, \quad \Psi(\widetilde{\mathfrak{p}}) := \left\{a \in A \mid \frac{a}{1} \in \widetilde{\mathfrak{p}}\right\}.$$

Prove that Φ and Ψ are well-defined and inverse of each other. (**Hint**. You have to show that $S^{-1}\mathfrak{p}$ is a prime ideal of $S^{-1}A$ if $\mathfrak{p}$ is a prime ideal of A. Notice that $A/\mathfrak{p}$ is an integral domain, and so $\overline{S}^{-1}(A/\mathfrak{p})$ can be viewed as a subring of the field of fractions of $A/\mathfrak{p}$. On the other hand, $S^{-1}A/S^{-1}\mathfrak{p} \simeq \overline{S}^{-1}(A/\mathfrak{p})$. Next you have to show that if $\frac{x}{1} \in S^{-1}\mathfrak{p}$ for some prime ideal $\mathfrak{p}$, then $x \in \mathfrak{p}$.)

(**Remark**. This problem implies that $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ is a bijection between prime ideals of A that do not intersect S and prime ideals of $S^{-1}A$.)

6. Suppose A is a unital commutative ring and $\mathfrak{m}$ is an ideal of A.

   (a) Prove that $\mathrm{Max}(A) = \{\mathfrak{m}\}$ if and only if $A^{\times} = A \setminus \mathfrak{m}$. (Recall that $A^{\times}$ is the group of units of A. A ring with only one maximal ideal is called a *local ring*.)

   (b) Prove that for all $\mathfrak{p} \in \mathrm{Spec}(A)$, $A_{\mathfrak{p}}$ is a local ring.

7. Suppose A is a unital commutative ring. Suppose $f_1, \ldots, f_n \in A \setminus \mathrm{Nil}(A)$ and $\langle f_1, \ldots, f_n \rangle = A$. Let $S_{f_i} := \{1, f_i, f_i^2, \ldots\}$.

   (a) Suppose $\mathfrak{b}$ is an ideal of A and for all i there exists a positive integer m_i such that $f_i^{m_i} \in \mathfrak{b}$. Prove that $\mathfrak{b} = A$. (**Hint**. Consider the radical of $\mathfrak{b}$.)

(b) Suppose M is an A-module and N is a submodule of M. Suppose $S_{f_i}^{-1}N = S_{f_i}^{-1}M$ for all i. Prove that $N = M$. (**Hint**. Suppose $x \in M$. Let $\mathfrak{b} := \{a \in A \mid ax \in N\}$. Prove that $\mathfrak{b}$ is an ideal of A and use the previous part to show $\mathfrak{b} = A$.)

(c) Suppose $S_{f_i}^{-1}M$ is a finitely generated $S_{f_i}^{-1}A$-module for all i. Prove that M is a finitely generated A-module. (**Hint**. For every i, use the assumption that $S_{f_i}^{-1}M$ is finitely generated to find $m_{i1}, \ldots, m_{ik_i} \in M$ such that $S_{f_i}^{-1}M$ is generated by $\frac{m_{ij}}{1}$'s. Let N be the submodule of M which is generated by m_{ij}'s. Use the previous part.)

(d) Suppose $S_{f_i}^{-1}A$ is a Noetherian ring for all i. Prove that A is Noetherian.

8. Suppose $\{M_i\}_{i\in I}$ is a family of A-modules and N is an A-module. Prove that

   (a) $\mathrm{Hom}_A(\oplus_{i\in I} M_i, N) \simeq \prod_{i\in I} \mathrm{Hom}_A(M_i, N)$,

   (b) $\mathrm{Hom}_A(N, \prod_{i\in I} M_i) \simeq \prod_{i\in I} \mathrm{Hom}_A(N, M_i)$

   as abelian groups.

9. Suppose $\{A_i\}_{i\in I}$ is a family of pairwise commuting (this means $A_iA_j = A_jA_i$ for every i, j) diagonalizable elements of $\mathrm{M}_n(k)$ where k is a field. Prove that A_i's are simultaneously diagonalizable; that means there exists $g \in \mathrm{GL}_n(k)$ such that gA_ig^{-1} is a diagonal matrix for every i.

   (**Hint**. Notice that the claim is equivalent to finding a k-basis of k^n which consists of eigenvectors of A_i's; that means you have to show there exist $v_1, \ldots, v_n \in k^n$ such that (1) $k^n = \bigoplus_i kv_i$ and (2) for all i, j, $A_jv_i \in kv_i$. To show this, suppose λ_i's are distinct eigenvalues of A_1. Show

$$k^n = \bigoplus_{i=1}^{m} \ker(A - \lambda_i I), \;\; A_j(\ker(A - \lambda_i I)) \subseteq \ker(A - \lambda_i I);$$

   and prove the claim by induction on the dimension of the underline vector space.)