1 Homework 8.

1. Suppose A is a local unital commutative ring and $\mathfrak{a}$ is an ideal of A.

   (a) Suppose M is a flat A-module. Prove that $\mathfrak{a} \otimes_A M \simeq \mathfrak{a}M$.

   (b) Suppose $0 \to M_1 \xrightarrow{i} M_2$ is injective, and M_1 and M_2 are flat A-modules. Prove that

   $$\mathrm{id}_{\mathfrak{a}} \otimes i : \mathfrak{a} \otimes_A M_1 \to \mathfrak{a} \otimes_A M_2$$

   is injective.

   (c) Suppose $0 \to N_1 \xrightarrow{i} N_2 \to N_3 \to 0$ is a SES, and N_2 and N_3 are flat A-modules. Prove that

   $$\mathfrak{a}N_2 \cap i(N_1) = i(\mathfrak{a}N_1).$$

   (**Hint.** Use problem 4 in HW 7: deduce that N_1 is a flat A-module.)

   (d) Suppose $M := F/K$ where F is a free A-module and K is a submodule of F. Suppose M is a flat A-module. Prove that

   $$\mathfrak{a}F \cap K = \mathfrak{a}K.$$

2. Suppose A is a local unital commutative ring and $\mathrm{Max}(A) = \{\mathfrak{m}\}$.

   (a) Suppose K is a finitely generated submodule of A^n and $\mathfrak{m}^n \cap K = \mathfrak{m}K$. Prove that K is a free A-module and $A^n = K \oplus N$ for some finitely generated submodule N of A^n.

   (b) Suppose M is a finitely presented A-module; that means, for some positive integer n, there is a finitely generated submodule K of A^n such that $M \simeq A^n/K$. Suppose M is a flat A-module. Prove that M is free.

   (**Hint.** (a) Notice that $0 \to \frac{K}{\mathfrak{m}K} \to \frac{A^n}{\mathfrak{m}^n}$ is injective of $(A/\mathfrak{m})$-vector spaces. Hence there are $x_1, \ldots, x_m \in K$ and $x_{m+1}, \ldots, x_n \in A^n$ such that $\overline{x}_i := x_i + \mathfrak{m}K$, for $i = 1..m$ is a $(A/\mathfrak{m})$-basis of $\frac{K}{\mathfrak{m}K}$ and $\overline{x}'_i := x_i + \mathfrak{m}^n$, for $i = 1..n$ is a $(A/\mathfrak{m})$-basis of $A^n/\mathfrak{m}^n$. Use Nakayama's lemma and show that

   $$K = \bigoplus_{i=1}^{m} Ax_i \quad \text{and} \quad A^n = \bigoplus_{i=1}^{n} Ax_i.$$

(b) Use Problem 1(d) and deduce that $\mathfrak{m}^n \cap K = \mathfrak{m}K$. Use part (a) and complete the proof.)

3. Suppose A is a unital commutative ring and M is a finitely presented flat A-module. Prove that for every $\mathfrak{p} \in \mathrm{Spec}(A)$, $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module. (**Remark.** This shows that every finitely presented flat module is locally free. Earlier you have seen that a finitely generated projective module is locally free. The converse of these statements are correct as well, and so for a finitely presented module we have

$$\text{flat} \iff \text{locally free} \iff \text{projective.))}$$

4. Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}$ as $\mathbb{C}$-algebras.

5. Let $A_p := \mathbb{Z}[x]/\langle x^2 + x + 1\rangle \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

   (a) Prove that A_p is a field if and only if $p \not\equiv 1 \pmod 3$ and $p \neq 3$.

   (b) Prove that $A_p \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ as rings if and only if $p \equiv 1 \pmod 3$.

   (c) Prove that A_p has a non-zero nilpotent element if and only if $p = 3$.