1 Homework 7.

1. Suppose $\{M_i\}_{i\in I}$ is a family of A-modules and N is an A-module. Prove that

   (a) $\mathrm{Hom}_A(\oplus_{i\in I} M_i, N) \simeq \prod_{i\in I} \mathrm{Hom}_A(M_i, N)$,

   (b) $\mathrm{Hom}_A(N, \prod_{i\in I} M_i) \simeq \prod_{i\in I} \mathrm{Hom}_A(N, M_i)$

   as abelian groups.

2. Suppose $\{A_i\}_{i\in I}$ is a family of pairwise commuting (this means $A_iA_j = A_jA_i$ for every i, j) diagonalizable elements of $\mathrm{M}_n(k)$ where k is a field. Prove that A_i's are simultaneously diagonalizable; that means there exists $g \in \mathrm{GL}_n(k)$ such that gA_ig^{-1} is a diagonal matrix for every i.

   (Hint: Notice that the claim is equivalent to finding a k-basis of k^n which consists of eigenvectors of A_i's; that means you have to show there exist $v_1, \ldots, v_n \in k^n$ such that (1) $k^n = \bigoplus_i kv_i$ and (2) for all i, j, $A_jv_i \in kv_i$. To show this, suppose λ_i's are distinct eigenvalues of A_1. Show

   $$k^n = \bigoplus_{i=1}^m \ker(A - \lambda_i I), \quad A_j(\ker(A - \lambda_i I)) \subseteq \ker(A - \lambda_i I);$$

   and prove the claim by induction on the dimension of the underline vector space.)

3. In this problem, you see the differences between a direct product and a direct sum. Among other things, you see that an infinite direct product is not necessarily a free module.

   (a) Let $\phi \in \mathrm{Hom}(\prod_{i=1}^\infty \mathbb{Z}, \mathbb{Z})$; let $e_j \in \prod_{i=1}^\infty \mathbb{Z}$ be

   $$e_j(i) := 0 \text{ if } i \neq j \quad \text{ and } e_i(i) = 1.$$

   Suppose $\phi(e_j) = n_j \neq 0$ for every j. Choose a sequence of positive integers $1 =: k_1 < k_2 < \cdots$ such that

   $$k_{j+1} \nmid k_j! n_j. \tag{1}$$

   Consider

   $$\Sigma := \{(a_i)_{i=1}^\infty |\, a_i \in \{0, k_i!\}\}. \tag{2}$$

(a-1) Argue why there exist two distinct elements $(a_i)_{i=1}^{\infty}$ and $(a'_i)_{i=1}^{\infty}$ of Σ such that

$$\phi((a_i)_{i=1}^{\infty}) = \phi((a'_i)_{i=1}^{\infty}). \tag{3}$$

(**Hint**. Notice that Σ is uncountble and $\mathbb{Z}$ is countable.)

(a-2) In the setting of the previous step, suppose i_0 is the first index where $a_{i_0} \neq a'_{i_0}$. Show that

$$\phi((a_{i_0} - a'_{i_0})e_{i_0}) \notin k_{i_0+1}\mathbb{Z}, \qquad \text{(Hint: use (1))}$$

and

$$\phi((a_{i_0} - a'_{i_0})e_{i_0}) \in k_{i_0+1}\mathbb{Z}; \qquad \text{(Hint: use (2) and (3))}$$

and get a contradiction.

(b) Use part (a) to deduce

$$\begin{aligned} \operatorname{Hom}(\textstyle\prod_{i=1}^{\infty} \mathbb{Z}, \mathbb{Z}) \to & \bigoplus_{i=1}^{\infty} \mathbb{Z}, \\ \phi \mapsto & (\phi(e_i))_{i=1}^{\infty} \end{aligned}$$

is an isomorphism.

(**Hint**. Suppose $\bigoplus_{i=1}^{\infty} \mathbb{Z} \subseteq \ker \phi$; then show

$$p^n | \phi(pa_1, p^2a_2, p^3a_3, \ldots)$$

for every n and deduce that $\phi(pa_1, p^2a_2, p^3a_3, \ldots) = 0$; observe that every element $(b_1, b_2, \ldots)$ can be written as a sum of two elements of the form $(2a_1, 2^2a_2, \ldots)$ and $(3a_1, 3^2a_2, \ldots)$.)

(c) Use part (b) to show $\prod_{i=1}^{\infty} \mathbb{Z}$ is not a free abelian group.

(d) Use part (b) to show

$$\operatorname{Hom}\left(\frac{\prod_{i=1}^{\infty} \mathbb{Z}}{\bigoplus_{i=1}^{\infty} \mathbb{Z}}, \mathbb{Z}\right) = 0.$$

4. Suppose A is an integral domain. Show that a submodule of a finitely generated free A-module is a free A-module if and only if A is a PID.

5. Suppose $(A, \mathfrak{m})$ is a <u>local</u> unital commutative ring; that means $\operatorname{Max}(A) = \{\mathfrak{m}\}$.

(a) (Nakayama's lemma) Suppose M is a finitely generated A-module. Suppose $M = \mathfrak{m}M$ where

$$\mathfrak{m}M = \{\sum_{i=1}^{n} a_i x_i |\ a_i \in \mathfrak{m}, x_i \in M\}.$$

Prove that $M = 0$.

(Hint: Let $y_1, \ldots, y_d$ be a generating set of M. By assumption, $\exists a_{ij} \in \mathfrak{m}$ such that

$$y_i = \sum_{j=1}^{d} a_{ij} y_j.$$

Hence $(I - [a_{ij}]) \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix} = 0$. Show that $I - [a_{ij}] \in \mathrm{GL}_d(A)$; and deduce $y_i = 0$; and so $M = 0$.)

(b) Suppose M is a finitely generated A-module. Prove that

$$d(M) = \dim_{\frac{A}{\mathfrak{m}}} \left(\frac{M}{\mathfrak{m}M} \right),$$

where $\frac{M}{\mathfrak{m}M}$ is viewed as a vector space over $\frac{A}{\mathfrak{m}}$.

(Hint: It is clear that $d(M) \geq \dim_{\frac{A}{\mathfrak{m}}}(\frac{M}{\mathfrak{m}M})$; now suppose

$$y_1 + \mathfrak{m}M, \ldots, y_d + \mathfrak{m}M$$

is an $\frac{A}{\mathfrak{m}}$-basis of $\frac{M}{\mathfrak{m}M}$, and let N be the submodule of M that is generated by y_i's. Use part (a) for $\frac{M}{N}$.)

(c) (f.g. projective $\Rightarrow$ locally free) Suppose P is a finitely generated projective A-module. Prove that P is free.

(Hint: Suppose $d(P) = d$; so there is a S.E.S.

$$0 \to N \to A^d \to P \to 0.$$

Since P is projective, we have that there is an A-module isomorphism $\phi : A^d \xrightarrow{\sim} P \oplus N$. Show that $\phi(\mathfrak{m}A^d) = \mathfrak{m}P \oplus \mathfrak{m}N$; and then use part (b).)

(Remark. This exercise implies that for an arbitrary unital commutative ring A, a finitely generated projective module P is locally free; that means for every $\mathfrak{p} \in \operatorname{Spec}(A)$, $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module. The converse of this statement is true as well: a f.g. locally free module is projective.)

6. Suppose $\{f_i\}_{i\in I} \subseteq \mathbb{Z}[x_1, \ldots, x_n]$ is a family of polynomials. For every unital commutative ring A, let

$$F(A) := \{(a_1, \ldots, a_n) \in A^n \mid \forall i \in I, f_i(a_1, \ldots, a_n) = 0\}.$$

   (a) Prove that F defines a functor from the category of unital commutative rings to the category of sets.

   (b) Prove that there exists a natural isomorphism from F to a representable functor.

   (**Hint.** Let $\mathfrak{a} := \langle f_i \mid i \in I\rangle$ and $R_0 := \mathbb{Z}[x_1, \ldots, x_n]/\mathfrak{a}$. Show that the following is a *natural bijection* between $\operatorname{Hom}_{\mathrm{Rng}}(R_0, A)$ and $F(A)$:

$$\phi \mapsto (\phi(x_1 + \mathfrak{a}), \ldots, \phi(x_n + \mathfrak{a})).$$

   Notice that the inverse of this map is given by the evaluation maps; for every $\mathbf{a} \in F(A)$, let

$$\phi_{\mathbf{a}}(f(x) + \mathfrak{a}) := f(\mathbf{a})$$

   and argue why this is well-defined.)

7. Suppose P and P' are projective A-modules, and

$$0 \to K \to P \xrightarrow{f} M \to 0$$

   and

$$0 \to K' \to P' \xrightarrow{f'} M \to 0$$

   are short exact sequences of A-modules. Prove that

$$P \oplus K' \simeq P' \oplus K.$$

   Hint: Let $L := \{(x, x') \in P \oplus P' | \, f(x) = f'(x')\}$. Show that L is a submodule of $P \oplus P'$. Notice that the following diagram is commuting and each row and column is an exact sequence; and then use the assumption that P and P' are projective to deduce $L \simeq P \oplus K'$ and $L \simeq P' \oplus K$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \ker \pi' & \xrightarrow{\sim} & K' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker \pi & \longrightarrow & L & \xrightarrow{\pi'} & P' & \longrightarrow & 0 \\
 & & \downarrow \wr & & \downarrow \pi & & \downarrow f' & & \\
0 & \longrightarrow & K & \longrightarrow & P & \xrightarrow{f} & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$