1 Homework 8.

1. (Yoneda's lemma) Suppose $\underline{C}$ is a locally small category, $a \in \mathrm{Obj}(\underline{C})$,

$$h_a : \underline{C} \to \underline{\mathrm{Set}}$$

is a representable functor, and $F : \underline{C} \to \underline{\mathrm{Set}}$ is a functor. Let $\mathrm{Nat}(h_a, F)$ be the class of all natural transformations from h_a to F.

   (a) Prove that the following is a bijection:

$$\theta_a : \mathrm{Nat}(h_a, F) \to F(a), \quad \theta(\eta) := \eta_a(1_a).$$

   (b) For $f \in \mathrm{Hom}_{\underline{C}}(a, a')$, let $\widehat{f} : h'_a \to h_a$ be

$$\widehat{f}_b(a' \xrightarrow{g} b) := a \xrightarrow{g \circ f} b.$$

   Prove that $\widehat{f}$ is a natural transformation.

   (c) Prove that θ_a is a natural bijection; that means, if $f \in \mathrm{Hom}_{\underline{C}}(a, a')$, then the following is a commuting diagram

$$\begin{array}{ccc} \mathrm{Nat}(h_a, F) & \xrightarrow{\theta_a} & F(a) \\ {\scriptstyle \psi(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \mathrm{Nat}(h_{a'}, F) & \xrightarrow{\theta_{a'}} & F(a') \end{array}$$

   where for every $b \in \mathrm{Obj}(\underline{C})$,

$$\psi(f)(\eta) := \eta \circ \widehat{f}.$$

2. Suppose D is a local Noetherian integral domain.

   (a) Prove that every submodule of a finitely generated projective D-module is projective if and only if D is a PID.

   (b) Find a local Noetherian integral domain which is not a PID.

   (c) Show that a submodule of a finitely generated projective module is not necessarily projective.

   (**Hint.** Use two results from last week (1) submodule of a finitely generated free D-module is free if and only if D is a PID, (2) finitely generated projective modules of a local Noetherian ring are free.)

3. Suppose A is a unital commutative ring, and M is an A-module. Let

$$T_M : \underline{A\text{--mod}} \to \underline{A\text{--mod}},\ \ T_M(N) := M \otimes_A N \quad \text{and} \quad T_M(f) := \mathrm{id}_M \otimes f,$$

for $f \in \mathrm{Hom}_A(N, N')$.

(a) Prove that T_M is a functor.

(b) Prove that there exists a natural isomorphism between $T_{M_1} \circ T_{M_2}$ and $T_{M_1 \otimes_A M_2}$.

(**Hint**. For $x_1 \in M_1$, let

$$f_{x_1} : M_2 \times N \to (M_1 \otimes_A M_2) \otimes N, \quad f_{x_1}(x_2, y) := (x_1 \otimes x_2) \otimes y.$$

Prove that f_{x_1} is A-bilinear. Deduce that there exists an A-module homomorphism

$$\phi_{x_1} : M_2 \otimes N \to (M_1 \otimes_A M_2) \otimes N,$$

such that

$$\phi_{x_1}(x_2 \otimes y) = (x_1 \otimes x_2) \otimes y.$$

Let

$$f : M_1 \times (M_2 \otimes N) \to (M_1 \otimes_A M_2) \otimes N, \quad f(x_1, z) := \phi_{x_1}(z).$$

Prove that f is A-bilinear. Deduce that there exists an A-module homomorphism

$$\phi : M_1 \otimes_A (M_2 \otimes_A N) \to (M_1 \otimes_A M_2) \otimes N,$$

such that

$$\phi(x_1 \otimes z) = f(x_1, z);$$

in particular, for every $x_1 \in M_1$, $x_2 \in M_2$, and $y \in N$,

$$\phi(x_1 \otimes (x_2 \otimes y)) = (x_1 \otimes x_2) \otimes y.$$

Similarly there exists an A-module homomorphism

$$\psi : (M_1 \otimes_A M_2) \otimes N \to M_1 \otimes_A (M_2 \otimes_A N),$$

such that

$$\psi((x_1 \otimes x_2) \otimes y) = x_1 \otimes (x_2 \otimes y).$$

Deduce that ϕ and ψ are inverse of each other. Convince yourself that this is a natural isomorphism. You do not need to include that in your solution.)

(**Remark**. We say that $M_1 \otimes_A (M_2 \otimes_A N)$ and $(M_1 \otimes_A M_2) \otimes_A N$ are naturally isomorphic.)

4. Suppose A is a unital commutative ring and M, N_1, and N_2 are A-modules.

   (a) Prove that there exists an A-module isomorphism

$$\phi : M \otimes_A (N_1 \oplus N_2) \to (M \otimes_A N_1) \oplus (M \otimes_A N_2),$$

   such that $\phi(x \otimes (y_1, y_2)) = (x \otimes y_1, x \otimes y_2)$.

   (b) Prove that the following is a splitting SES

$$0 \to M \otimes_A N_1 \xrightarrow{\mathrm{id}_M \otimes j_1} M \otimes_A (N_1 \oplus N_2) \xrightarrow{\mathrm{id}_M \otimes p_2} M \otimes_A N_2 \to 0,$$

   where $j_1 : N_1 \to N_1 \oplus N_2, \quad j_1(x_1) := (x_1, 0)$ and

$$p_2 : N_1 \oplus N_2 \to N_2, \quad p_2(x_1, x_2) := x_2.$$

(**Hint.** Let

$f : M \times (N_1 \oplus N_2) \to (M \otimes_A N_1) \oplus (M \otimes_A N_2), \;\; f(x, (y_1, y_2)) := (x \otimes y_1, x \otimes y_2).$

Notice that f is an A-bilinear map. Deduce that there exists an A-module homomorphism, as given in the statement of the problem. Let

$$\psi : (M \otimes_A N_1) \oplus (M \otimes_A N_2) \to M \otimes_A (N_1 \oplus N_2),$$

$$\psi(z_1, z_2) := (\mathrm{id}_M \otimes j_1)(z_1) + (\mathrm{id}_M \otimes j_2)(z_2).$$

Then ψ is an A-module homomorphism. Check that

$$\psi(\phi(x \otimes (y_1, y_2))) = x \otimes (y_1, y_2),$$

and deduce that $\psi \circ \phi$ is identity. Similarly, you can obtain that $\phi \circ \psi$ is identity.

Use these isomorphisms, to show that the given sequence is isomorphic to the splitting SES

$$0 \to M \otimes_A N_1 \to (M \otimes_A N_1) \oplus (M \otimes_A N_2) \to M \otimes_A N_2 \to 0.)$$

5. (You do not have to write anything for this problem; only justify and understand all the statements. This extends the result of the previous problem, and it is a useful result to have in your toolbox.)

   For two functors F_1 and F_2, we say $F_1 \simeq F_2$ if there exists a natural isomorphism $\eta : F_1 \to F_2$. Suppose $\{M_i\}_{i\in I}$ is a family of A-modules and N is an A-module.

   (a) Suppose $\{F_i\}_{i\in I}$ is a family of functors from $\underline{A\text{-mod}}$ to itself. Define the functor $\prod_{i\in I} F_i$.

   (b) Prove that
   $$\prod_{i\in I} h_{M_i} \simeq h_{\bigoplus_{i\in I} M_i}.$$

   (c) Justify why we have
   $$\begin{aligned} h_{\bigoplus_{i\in I} N\otimes_A M_i} \simeq &\prod_{i\in I} h_{N\otimes_A M_i} \simeq \prod_{i\in I}(h_{M_i}\circ h_N) \\ \simeq &(\prod_{i\in I} h_{M_i})\circ h_N \simeq h_{\bigoplus_{i\in I} M_i}\circ h_N \\ \simeq &h_{N\otimes_A(\bigoplus_{i\in I} M_i)}. \end{aligned}$$

   (d) Prove that $\bigoplus_{i\in I}(N\otimes_A M_i) \simeq N\otimes_A (\bigoplus_{i\in I} M_i)$ as A-modules.

6. Suppose A is a local unital commutative ring and K is a field.

   (a) Suppose V and W are two K-vector spaces. Prove that
   $$\dim_K(V\otimes_K W) = (\dim_K V)(\dim_K W)$$

   (**Hint**. Use problem 5.)

   (b) Suppose M and N are finitely generated A-modules, and $M\otimes_A N = 0$. Prove that either $M = 0$ or $N = 0$.

   (Hint. Suppose $\operatorname{Max}(A) = \{\mathfrak{m}\}$. Let $k := A/\mathfrak{m}$. Argue
   $$M/\mathfrak{m}M \simeq M\otimes_A k \quad \text{and} \quad N/\mathfrak{m}N \simeq N\otimes_A k.$$
   Show that $(M/\mathfrak{m}M)\otimes_k (N/\mathfrak{m}N) = 0$. Use Nakayama's lemma.)

(**Remark**. Notice that $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$ and so it is crucial that A is local. For an arbitrary ring A, we deduce that $M \otimes_A N = 0$ implies for any $\mathfrak{p} \in \operatorname{Spec} A$ either $M_{\mathfrak{p}} = 0$ or $N_{\mathfrak{p}} = 0$.)

7. Suppose A is a unital commutative ring, $S \subseteq A$ is a multiplicatively closed subset, and M is an A-module.

   (a) Convince yourself that localizing defines an exact functor from $\underline{A\text{-mod}}$ to $\underline{S^{-1}A\text{-mod}}$. (You do not need to write any argument for this part.)

   (b) Prove that $S^{-1}A \otimes_A M \simeq S^{-1}M$; deduce that $S^{-1}A$ is a flat A-module.

   (c) Prove that, if M is a flat A-module, then $S^{-1}M$ is a flat $S^{-1}A$-module.

   (d) Prove that $\frac{x_1 \otimes x_2}{1} \mapsto \frac{x_1}{1} \otimes \frac{x_2}{1}$ gives us a well-defined $S^{-1}A$-module isomorphism

   $$S^{-1}(M_1 \otimes_A M_2) \xrightarrow{\sim} S^{-1}M_1 \otimes_{S^{-1}A} S^{-1}M_2.$$

   (e) Prove that, if $M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module for every $\mathfrak{p} \in \operatorname{Spec}(A)$, then M is flat. (Hint: look at HW5, localizing a module.)