Lecture 04: Fiber over a prime

Friday, April 6, 2018 9:58 AM

We were proving the following proposition:

<u>Proposition</u>. Let $f: A \to B$ be a ring homomorphism. Then for any $\mathfrak{p} \in \mathrm{Spec}(A)$, there is a bijection between $(f^*)^{-1}(\mathfrak{p})$ and $\mathrm{Spec}(B \otimes_A k(\mathfrak{p}))$ where $k(\mathfrak{p}) = A_{\mathfrak{p}} / \mathfrak{p} A_{\mathfrak{p}}$.

<u>Pf</u>. So far we have proved, $B \otimes_A k(\mathfrak{p}) \simeq f(S_{\mathfrak{p}})^{-1} B / f(S_{\mathfrak{p}})^{-1} \langle f(\mathfrak{p}) \rangle$.

$$B \xrightarrow{\theta} f(S_{\mathfrak{p}})^{-1} B \xrightarrow{\pi} f(S_{\mathfrak{p}})^{-1} B / f(S_{\mathfrak{p}})^{-1} \langle f(\mathfrak{p}) \rangle$$

$$\xleftarrow[\sim]{\theta^*} \mathrm{Spec}(f(S_{\mathfrak{p}})^{-1} B) \supseteq V(f(S_{\mathfrak{p}})^{-1} \langle f(\mathfrak{p}) \rangle) \xleftarrow[\sim]{\pi^*} \mathrm{Spec}(\ldots)$$

$$\{\mathfrak{q} \in \mathrm{Spec}(B) \mid \mathfrak{q} \cap f(S_{\mathfrak{p}}) = \emptyset\}$$

Hence $\theta^* \circ \pi^*$ is injective, and

$$\mathrm{Im}(\theta^* \circ \pi^*) = \{\mathfrak{q} \in \mathrm{Spec}(B) \mid \mathfrak{q} \cap f(S_{\mathfrak{p}}) = \emptyset, \; f(S_{\mathfrak{p}})^{-1} \mathfrak{q} \supseteq f(S_{\mathfrak{p}})^{-1} \langle f(\mathfrak{p}) \rangle \}$$

<u>Claim 3</u> $\mathfrak{q} \in \mathrm{Im}(\theta^* \circ \pi^*) \Leftrightarrow f^*(\mathfrak{q}) = \mathfrak{p}$.

<u>Pf</u>. $(\Rightarrow)$. $\cdot$ $\mathfrak{q} \cap f(S_{\mathfrak{p}}) = \emptyset \Rightarrow f^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}$

$\cdot$ $\mathfrak{q} = (f(S_{\mathfrak{p}})^{-1} \mathfrak{q})^c \supseteq f(\mathfrak{p}) \Rightarrow f^{-1}(\mathfrak{q}) \supseteq \mathfrak{p}$

$\Rightarrow f^*(\mathfrak{q}) = \mathfrak{p}$.

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($\Leftarrow$). $f^*(\mathfrak{q})=\mathfrak{p} \Rightarrow \mathfrak{q}\cap f(S_{\mathfrak{p}})=\emptyset$.

. $f(\mathfrak{p})\subseteq \mathfrak{q} \Rightarrow \langle f(\mathfrak{p})\rangle \subseteq \mathfrak{q} \Rightarrow f(S_{\mathfrak{p}})^{-1}\mathfrak{q} \supseteq f(S_{\mathfrak{p}})^{-1}\langle f(\mathfrak{p})\rangle$.

And so $\theta^*\circ\pi^*: \operatorname{Spec}\left(f(S_{\mathfrak{p}})^{-1}B/f(S_{\mathfrak{p}})^{-1}\langle f(\mathfrak{p})\rangle\right) \longrightarrow \operatorname{Spec}(B)$

is injective and $\operatorname{Im}(\theta^*\circ\pi^*)=(f^*)^{-1}(\mathfrak{p})$. And claim follows. ■

<u>Remark 1</u> Careful examination of the above proof shows that the above bijection is a homeomorphism (with induced topology on $(f^*)^{-1}(\mathfrak{p})$.)

<u>Remark 2.</u> The above given bijection preserves inclusion; in general if $f: A\to B$ is a ring homomorphism, $\mathfrak{q}_1\subseteq\mathfrak{q}_2\in\operatorname{Spec}(B)$, then $f^*(\mathfrak{q}_1)\subseteq f^*(\mathfrak{q}_2)$. And so length of a longest chain of prime ideals of $B\otimes_A k(\mathfrak{p})$ is the same as the length of a longest chain of prime ideals of $(f^*)^{-1}(\mathfrak{p})$.

<u>Corollary.</u> $\mathfrak{p}\in\operatorname{Im} f^* \Leftrightarrow \mathfrak{p}^{ec}=\mathfrak{p}$.

<u>Pf.</u> $(f^*)^{-1}(\mathfrak{p})\neq\emptyset \Leftrightarrow B\otimes_A k(\mathfrak{p})\neq 0 \Leftrightarrow f(S_{\mathfrak{p}})^{-1}\langle f(\mathfrak{p})\rangle \subsetneq f(S_{\mathfrak{p}})^{-1}B$

$\Leftrightarrow f(S_{\mathfrak{p}})\cap\mathfrak{p}^e=\emptyset \Leftrightarrow \mathfrak{p}^{ec}\subseteq\mathfrak{p} \Leftrightarrow \mathfrak{p}^{ec}=\mathfrak{p}$. ■

Lecture 04: Nakayama's lemma

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Another result that you learned in 200 B as part of your HW assignments was a special case of Nakayama's lemma. Now we are going to treat the general case (commutative version).

Proposition. Suppose M is a finitely generated A-module. Let $\mathfrak{a} \lhd A$, and $\phi \in \mathrm{End}_A(M)$. If $\phi(M) \subseteq \mathfrak{a} M$, then

$$\phi^n + a_1 \phi^{n-1} + \cdots + a_n = 0$$

for some $a_i \in \mathfrak{a}$.

Pf. Let $\widetilde{R} := \mathrm{End}_A(M)$. Then $\widetilde{R}$ is a ring (not necessarily commutative). Let $\overline{A} \subseteq \mathrm{End}_A(M)$ be the subring given by $\ell_a : M \to M$, $\ell_a(m) := am$. Then $\overline{A}[\phi]$ is a commutative subring of $\mathrm{End}_A(M)$. Suppose $m_1, \ldots, m_n$ is a generating set of M. Then, by $\phi(M) \subseteq \mathfrak{a} M$, $\phi(m_i) = \sum_{j=1}^{n} a_{ij} m_j$ for some $a_{ij} \in \mathfrak{a}$. Hence

$$\left(\phi I_n - \begin{bmatrix} \ell_{a_{11}} & \cdots & \ell_{a_{1n}} \\ \vdots & & \vdots \\ \ell_{a_{n1}} & \cdots & \ell_{a_{nn}} \end{bmatrix} \right) \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} \phi(m_1) \\ \vdots \\ \phi(m_n) \end{bmatrix} - \begin{bmatrix} \sum a_{1j} m_j \\ \vdots \\ \sum a_{nj} m_j \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

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And so
$$\underbrace{\begin{bmatrix} \phi-\bar{a}_{11} & \cdots & -\bar{a}_{1n} \\ \vdots & \ddots & \vdots \\ -\bar{a}_{n1} & \cdots & \phi-\bar{a}_{nn} \end{bmatrix}}_{\text{a matrix with entries in a commutative ring.}} \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

After multiplying both sides by its adjoint matrix we get:

$$\det(\phi I - [\bar{a}_{ij}]) \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$\Rightarrow$ $\det(\phi I - [\bar{a}_{ij}]) = 0$ in $\mathrm{End}_A(M)$ as m_i's generate M.

But $\det(xI - [a_{ij}]) = x^n \pmod{\mathfrak{a}}$; and so

$$0 = \det(\phi I - [\bar{a}_{ij}]) = \phi^n + \bar{a}_{n-1}\phi^{n-1} + \cdots + \bar{a}_0$$

for some $a_i \in \mathfrak{a}$. ∎

Corollary. $\left.\begin{array}{l} M: \text{finitely gen. } A\text{-module} \\ \mathfrak{a} \lhd A \\ \mathfrak{a}M = M \end{array}\right\} \Rightarrow \exists\, x \in A$ s.t. $xM = 0$ and $x \equiv 1 \pmod{\mathfrak{a}}$.

Pf. Let $\phi = \mathrm{id}_M \in \mathrm{End}_A(M)$. Then $\phi^n + \bar{a}_{n-1}\phi^{n-1} + \cdots + \bar{a}_0 = 0$ for some $a_i \in \mathfrak{a}$; this implies $(1+a)M = 0$ for some $a \in \mathfrak{a}$. ∎

Nakayama's lemma. Suppose M is a finitely generated A-mod, and $J(A)M = M$. Then $M = 0$.

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Pf. By the previous corollary, $\exists\, a \in J(A)$ s.t. $(1+a)M = 0$.

As we have seen in the previous lecture, $1+a \in A^{\times}$. Hence $M=0$. ∎

We did not have time to go over this application of Nakayama's in lecture. Please go over this.

Corollary. Let $d_A(M)$ be the minimum number of generators of M. Then for a finitely generated A-mod M, we have

$$d_A(M) = d_{A/J(A)}\left(M/J(A)M\right).$$

Pf. Clearly $d_A(M) \geq d_{A/J(A)}\left(M/J(A)M\right)$. Now suppose $\{m_1 + J(A)M, \ldots, m_n + J(A)M\}$ is a set of generators of $M/J(A)M$. Let N be the A-mod generated by $m_1, \ldots, m_n$. Then $M = N + J(A)M$; and so $M/N = J(A)\, M/N$. Therefore by Nakayama's lemma $M/N = 0$, which implies $M=N$; and so

$$d_A(M) \leq n. \quad ∎$$

This is particularly strong when A is a local ring; that means $\mathrm{Max}(A) = \{\mathfrak{m}\}$. In this case, $J(A) = \mathfrak{m}$, and $A/J(A)$ is a field. And so, when $(A, \mathfrak{m})$ is a local ring, $\dim_{A/\mathfrak{m}}\left(M/\mathfrak{m}M\right) = d_A(M)$.