Lecture 06: Reduced primary decomposition

Thursday, April 12, 2018 10:24 PM

In the previous lecture we were proving:

<u>Lemma</u>. (1) Suppose $\mathfrak{q}$ and $\mathfrak{q}'$ are $\mathfrak{p}$-primary. Then $\mathfrak{q}\cap\mathfrak{q}'$ is $\mathfrak{p}$-primary.

(2) A decomposable ideal has a reduced primary decomposition.

<u>Pf</u>. It is clear that it is enough to prove (1).

- $\sqrt{\mathfrak{q}\cap\mathfrak{q}'} \subseteq \sqrt{\mathfrak{q}} = \mathfrak{p}$
- $x \in \mathfrak{p} \Rightarrow \exists\, n, n',\quad x^n \in \mathfrak{q}$ and $x^{n'} \in \mathfrak{q}' \Rightarrow x^{n+n'} \in \mathfrak{q}\mathfrak{q}' \subseteq \mathfrak{q}\cap\mathfrak{q}'$

$$\Rightarrow x \in \sqrt{\mathfrak{q}\cap\mathfrak{q}'}.$$

- $xy \in \mathfrak{q}\cap\mathfrak{q}'$ and $y \notin \sqrt{\mathfrak{q}\cap\mathfrak{q}'} = \mathfrak{p} \Rightarrow \begin{cases} xy\in\mathfrak{q} \text{ and } y\notin\sqrt{\mathfrak{q}} \\ xy\in\mathfrak{q}' \text{ and } y\notin\sqrt{\mathfrak{q}'} \end{cases}$

$$\Rightarrow \begin{cases} x\in\mathfrak{q} \\ x\in\mathfrak{q}' \end{cases} \Rightarrow x \in \mathfrak{q}\cap\mathfrak{q}'. \qquad \blacksquare$$

($\mathfrak{q}$ and $\mathfrak{q}'$ are primary)

<u>Theorem</u>. Suppose $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is a reduced primary decomposition.

Then $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} = \operatorname{Spec}(A) \cap \{\sqrt{(\mathfrak{a}:x)} \mid x\in A\}$; and

so it just depends on $\mathfrak{a}$.

<u>Pf</u>. ($\subseteq$) $\mathfrak{q}_i \not\supseteq \bigcap_{\substack{j=1\\ j\neq i}}^{n} \mathfrak{q}_j \Rightarrow \exists\; x_i \in \bigcap_{\substack{j=1\\ j\neq i}}^{n} \mathfrak{q}_j \setminus \mathfrak{q}_i$.

$$\Rightarrow (\mathfrak{a}:x_i) = \left(\bigcap_{j=1}^{n}\mathfrak{q}_j : x_i\right) = \bigcap_{j=1}^{n} (\mathfrak{q}_j : x_i)$$

Lecture 06: First uniqueness theorem

Tuesday, April 10, 2018 10:53 PM

$$\boxed{(\mathfrak{q}_j : x_i) = \begin{cases} A & \text{if } i \neq j \\ \mathfrak{p}_i\text{-primary} & \text{if } i = j \end{cases}}$$

$\Longrightarrow$ $(\mathfrak{a} : x_i) = (\mathfrak{q}_i : x_i)$ which is $\mathfrak{p}_i$-primary

$\Longrightarrow$ $\mathfrak{p}_i = \sqrt{(\mathfrak{a} : x_i)}$.

(2) $(\mathfrak{a} : x) = \bigcap_{j=1}^{n} (\mathfrak{q}_j : x)$

$$\Longrightarrow \sqrt{(\mathfrak{a} : x)} = \bigcap_{j=1}^{n} \sqrt{(\mathfrak{q}_j : x)} = \bigcap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j.$$

$$\boxed{\sqrt{(\mathfrak{q}_j : x)} = \begin{cases} A & x \in \mathfrak{q}_j \\ \mathfrak{p}_j & x \notin \mathfrak{q}_j \end{cases}}$$

Suppose $\mathfrak{p} = \sqrt{(\mathfrak{a} : x)} \in \operatorname{Spec}(A)$. Then $\mathfrak{p} = \bigcap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j$; and so $\exists\, j$ s.t. $\mathfrak{p} = \mathfrak{p}_j$. $\blacksquare$

<u>Def.</u> Suppose $\mathfrak{a} \trianglelefteq A$ is decomposable and $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is a primary decomposition of $\mathfrak{a}$. Then $\operatorname{Ass}(\mathfrak{a}) := \{\sqrt{\mathfrak{q}_i} \mid 1 \le i \le n\}$ is called the set of primes that are <u>associated</u> with $\mathfrak{a}$.

Lecture 06: Minimal prime ideals

Tuesday, April 10, 2018 11:12 PM

Proposition. Suppose $\mathfrak{a}$ is decomposable. Then

(0) $\mathrm{Ass}(\mathfrak{a}) \subseteq V(\mathfrak{a})$.

(1) $\forall \mathfrak{p} \in V(\mathfrak{a}), \exists \mathfrak{p}' \in \mathrm{Ass}(\mathfrak{a}), \mathfrak{p}' \subseteq \mathfrak{p}$; in particular

$\{$minimal elements of $V(\mathfrak{a})\} = \{$minimal elements of $\mathrm{Ass}(\mathfrak{a})\}$.

Pf (0) $\mathfrak{a} = \bigcap \mathfrak{q}_i \Rightarrow \mathfrak{q}_i \mid \mathfrak{a}$

$\mathfrak{p}_i = \sqrt{\mathfrak{q}_i} \mid \mathfrak{q}_i$ $\Big\} \Rightarrow \mathfrak{p}_i \mid \mathfrak{a} \Rightarrow \mathfrak{p}_i \in V(\mathfrak{a})$.

(1) $\mathfrak{p} \in V(\mathfrak{a}) \Rightarrow \bigcap \mathfrak{q}_i \subseteq \mathfrak{p} \Rightarrow \sqrt{\bigcap \mathfrak{q}_i} \subseteq \sqrt{\mathfrak{p}} = \mathfrak{p}$

$\Rightarrow \mathfrak{p} \mid \prod \mathfrak{p}_i \Rightarrow \mathfrak{p} \mid \mathfrak{p}_j$ for some j.

$\Rightarrow \mathfrak{p}_j \subseteq \mathfrak{p}$ for some j. $\blacksquare$

Proposition. Suppose $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ is a reduced primary decomposition, and $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. Then $\bigcup_{i=1}^n \mathfrak{p}_i = \{x \in A \mid (\mathfrak{a} : x) \neq \mathfrak{a}\}$.

In particular, if $D(A) := \{x \in A \mid \exists y \in A \setminus \{0\}, xy = 0\}$, then

$$D(A) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(0)} \mathfrak{p} \quad \text{if } 0 \text{ is decomposable.}$$

Pf. • $(\mathfrak{a} : x) \neq \mathfrak{a} \Rightarrow \bigcap_{x \notin \mathfrak{q}_j} (\mathfrak{q}_j : x) \neq \bigcap \mathfrak{q}_j$.

$\Rightarrow \exists j$ s.t. $(\mathfrak{q}_j : x) \neq \mathfrak{q}_j \Rightarrow \exists j$ s.t. $x \in \mathfrak{p}_j$.

Lecture 06: Zero-divisors

Tuesday, April 10, 2018 11:32 PM

- Suppose $x \in \mathfrak{p}_j$, for some j. Then $x \in \sqrt{(\mathfrak{a}:y)}$ for some y such that $\sqrt{(\mathfrak{a}:y)}$ is prime. $(*)$ And so

  $\exists n \in \mathbb{Z}^+$ s.t. $x^n . y \in \mathfrak{a}$.

  If $(\mathfrak{a}:x) = \mathfrak{a}$, then by induction on n, $y \in \mathfrak{a}$; this implies $(\mathfrak{a}:y) = A$ which contradicts $(*)$. ■

- <u>$D(A)$</u> is the set of zero-divisors $\cup \{0\}$. (We will consider 0 a zero-divisor as well.)

<u>Cor.</u> $D(A) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(0)} \mathfrak{p}$ and $\mathrm{Nil}(A) = \bigcap_{\substack{\mathfrak{p}:\text{ minimal} \\ \text{in } \mathrm{Ass}(0)}} \mathfrak{p}$.

As we have seen before, $f^*: \mathrm{Spec}(S^{-1}A) \hookrightarrow \mathrm{Spec}(A)$ and $\mathrm{Im}(f^*) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \cap S = \emptyset\}$. This is a good technique to focus on primes that are "co-prime" to some elements.

Lecture 06: Contraction and extension of primary ideals and ring of fractions

Wednesday, April 11, 2018 8:45 AM

<u>Proposition.</u> Let $f: A \to S^{-1}A$, $f(a) = \frac{a}{1}$. Then

(1) If $\mathfrak{q}$ is $\mathfrak{p}$-primary and $\mathfrak{p} \cap S \neq \emptyset$, then $S^{-1}\mathfrak{q} = S^{-1}A$.

(2) If $\mathfrak{q}$ is $\mathfrak{p}$-primary and $\mathfrak{p} \cap S = \emptyset$, then $S^{-1}\mathfrak{q}$ is $S^{-1}\mathfrak{p}$-primary.

(3) Any primary ideal of $S^{-1}A$ is of the form $S^{-1}\mathfrak{q}$ where $\mathfrak{q}$ is a $\mathfrak{p}$-primary and $\mathfrak{p} \cap S = \emptyset$.

(4) $$\left\{ \mathfrak{q} \lhd A \;\middle|\; \begin{array}{c} \mathfrak{q}: \mathfrak{p}\text{-primary} \\ \mathfrak{p} \cap S = \emptyset \end{array} \right\} \underset{c}{\overset{e}{\rightleftarrows}} \{ \tilde{\mathfrak{q}} \lhd S^{-1}A \mid \tilde{\mathfrak{q}} : \text{primary} \}$$

The above maps are inverse of each other.

<u>Pf.</u> (1) $\sqrt{\mathfrak{q}} = \mathfrak{p}$. If $s \in S \cap \mathfrak{p}$, then $\exists\, n \in \mathbb{Z}^+$, $s^n \in S \cap \mathfrak{q}$; and so $S^{-1}\mathfrak{q} = S^{-1}A$.

(we will continue next lecture.)